ANALYSIS QUALIFYING EXAM

Spring 2002
January 26, 2002, 9:00 am - 12:00 noon
Room 408 Carver

Instructions

- Write your complete social security number on every page that you turn in. Do NOT write your name on any sheet that you turn in.
- Work 6 problems, with at least 2 from Part I and at least 3 from Part II. No credit will be given for additional problems, and if additional problems are turned in, only the worst ones will be counted.
- Work each problem on a separate sheet of paper, and clearly indicate the part and problem number.
- To pass, you must receive substantial credit from each part. One correct problem will be counted more than two "half correct" problems in the grading.

Part I: Complex Analysis

1. Evaluate

$$\int_0^{2\pi} \frac{d\theta}{3+\cos(\theta)+2\sin(\theta)}$$

2. Suppose U is an open set containing the closed unit disk and f is analytic on U such that $f(0) = 2/3$, and $|f(z)| > 1$ when $|z| = 1$. Prove that f has a zero in the open unit disk.

3. How many zeros does $z^6 + 4z^2 - 1$ have in the annulus $\{z : 1 \leq |z| \leq 2\}$?

4. Suppose f is an analytic map from the open unit disk D into itself such that f has two fixed points in D. Prove that $f(z) = z$ for all $z \in D$.

Part II: Real Analysis

1. Given a Lebesgue measurable extended-real valued function f on $[0,1]$ which takes the values $\pm\infty$ only on a set of measure 0, and given $\epsilon > 0$, there is an M such that $|f| \leq M$ except on a set of measure less than ϵ.

2. Let g be an absolutely continuous monotone function on $[0,1]$, and E a set of Lebesgue measure 0. Then $g(E)$ has measure 0.

3. Find continuous functions $f_n : [0,1] \to [0,\infty)$ such that $f_n \to 0$ for all $x \in [0,1]$ as $n \to \infty$, $\int_0^1 f_n(x)\,dx \to 0$, but $\sup_n f_n$ is not in L_1. (This shows that the conclusion of the Dominated Convergence Theorem may hold even when part of its hypothesis is violated).

4. Let $(X, \mathcal{M}, \mu)$ be a measure space with finite signed measure μ. Show that for each $E \in \mathcal{M}$,

$$\sup_{A \subset E} |\mu(A)| \leq |\mu|(E) \leq 2 \sup_{A \subset E} |\mu(A)|.$$

5. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. If f is integrable, compute and justify the limit

$$\lim_{n\to\infty} \int_X |f(x)|^{1/n}\, d\mu(x).$$

6. Let (X, μ) be a measure space with $\mu(X) < \infty$. So for $f \in L^\infty(X)$, $f \in L^p(X)$ for all $p \geq 1$. Show

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p.$$