Qualifying Examination in Analysis

Spring 2001

• Write your student identification number on every page of the solutions you hand in. Do not write your name.

• Hand in a total of 6 problems, including at least 2 from each part. You will not get partial credit for attempting any more than 6 problems.

• To pass, you must get substantial credit from both Parts I and II.

Part I. Real Analysis

1. Let $(X, \mathcal{A}, \mu)$ be an arbitrary measure space with μ a positive measure. Recall that a measure space is σ-finite if X can be written as a countable union of sets of finite measure. Prove that $(X, \mathcal{A}, \mu)$ is σ-finite if and only if there exists a strictly positive function $f \in L^1(\mu)$.

2. Give an example of each of the following:

a) A function f which is unbounded but Lebesgue integrable on $(0, \infty)$.

b) A function f which is Lipschitz continuous but not differentiable everywhere.

c) A function f which is absolutely continuous but not Lipschitz continuous on $[0, 1]$.

d) A sequence $\{f_n\}$ of continuous functions on $[0, 1]$ that converges pointwise to a function f on $[0, 1]$, but f is not continuous.

e) A sequence $\{f_n\}$ of functions that converges to zero pointwise on $[0, 1]$ but not in $L^1([0, 1])$.

3. Let $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$ and $\Omega \subset \mathbb{R}^N$.

a) Show that if $f_n \to f$ in $L^p(\Omega)$ and $g_n \to g$ in $L^q(\Omega)$ then $f_n g_n \to fg$ in $L^1(\Omega)$.

b) Explain carefully what is meant by the statement that $L^q(\Omega)$ is the dual space of $L^p(\Omega)$.

4. If $f \in L^q(\mathbb{R}^N)$ for some $q < \infty$, show that

$$\lim_{p\to\infty} ||f||_{L^p} = ||f||_{L^\infty}$$

Also, show by example that the conclusion may be false without the assumption that $f \in L^q(\mathbb{R}^N)$.

5. Show that

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{x}{n+1}\right)$$

converges pointwise on $\mathbb{R}$ and uniformly on each bounded interval of $\mathbb{R}$ to a differentiable function f which satisfies $|f(x)| \leq |x|$.

6. Prove the Riemann-Lebesgue Lemma: For any $f \in L^1(\mathbb{R})$

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) e^{inx}\, dx = 0$$

You may use the fact that for any $f \in L^1(\mathbb{R})$ and any $\epsilon > 0$ there exists a step function $g \in L^1(\mathbb{R})$ such that $\int_{-\infty}^{\infty} |f(x) - g(x)|\, dx < \epsilon$.

Part II. Complex Analysis

1. Let $\Gamma = \{z \in \mathbb{C} : |z - (6+i)| = 3\}$. Evaluate $\displaystyle\oint_\Gamma (\bar{z} - i)^2 \, dz$ where the orientation of Γ is in the counterclockwise direction.

2. Assume that f, g are holomorphic in the disk $D = \{z \in \mathbb{C} : |z| < 1\}$, continuous on $\overline{D}$ and have no zeros in D. If $|f(z)| \equiv |g(z)|$ for $|z| = 1$, prove that $f(z) = kg(z)$ in D, for some constant k of modulus 1.

3. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$. (Suggestion: Integrate $\displaystyle\frac{\cot(\pi z)}{z^2+1}$ around a suitable contour.)

4. Prove that $\displaystyle F(z) = \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2}$ is defined and holomorphic for $z \neq \pm 1, \pm 2 \ldots$.