

ANALYSIS QUALIFYING EXAMINATION

Fall 1998

Saturday, August 15, 1998, 9:00am-12:00 noon

Room 408 Carver

Instructions:

- a. **Write your social security number on every page that you turn in.** Do NOT write your name on any sheet you turn in.
- b. Work no more than 6 problems, with no more than 4 from either Part I or Part II. No credit will be given for additional problems, and if additional problems are turned in, only the worst ones will be counted.
- c. Work each problem on a separate sheet of paper and clearly indicate the part and problem number.
- d. To pass you must receive substantial credit from each part. One correct problem will be counted more than two "half correct" problems in the grading.

PART I. Complex Analysis

1. Let γ be the circle of radius 2 centered at the origin and with counterclockwise orientation. Determine the value of each of the following:

a. $\int_{\gamma} (\bar{z})^n dz$ (n an integer) b. $\int_{\gamma} \frac{e^{\sin z}}{e^{\pi z} - 2} dz$

2. Describe geometrically the domain of analyticity (with principle branch of the logarithm) of the function $f(z) = \log \frac{z-i}{1-z}$.
3. Suppose $f(z)$ is analytic everywhere except at the origin, where it has a pole. Suppose $f(1/n) = n$ for $n = 1, 2, \dots$
 - a. Show that $f(z) = z^{-1}$.
 - b. Show this is not necessarily true if it is only assumed that f is analytic everywhere except at the origin. (Hint: First understand why f is not uniquely determined.)
4. Determine the supremum of $|f(1/2)|$ over the class of all functions f which are analytic in the unit disc and satisfy $|f(z)| \leq 4|z|^{1.3}$ in the unit disc. Find all functions f (if any) for which the supremum is attained.

PART II. Real Analysis

In all the problems below, \mathbf{R} denotes the real line, λ denotes Lebesgue measure on \mathbf{R} and \mathcal{L} denotes the Lebesgue measurable sets on \mathbf{R} .

1. Let (X, \mathcal{M}, μ) be a measure space and $f : X \rightarrow \mathbf{R}$. Suppose that $\{x : f(x) > \tau\}$ is measurable for each rational number τ . Is f measurable?
2. a. Give an example of a sequence $\{f_n\}_{n=1}^{\infty}$ of nonnegative functions on the interval $0 \leq x \leq 1$ that satisfies the following properties:
 - (i) f_n is continuous, for $n = 1, 2, 3, \dots$
 - (ii) $\{f_n(x)\}_{n=1}^{\infty}$ is unbounded for each x in $[0, 1]$.
 - (iii) $\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = 0$.
 b. Can a sequence satisfying (i)-(iii) also satisfy (iv) below?
 (iv) $\liminf_{n \rightarrow \infty} f_n(x) > 0$, for all $x \in E$, for some $E \subset [0, 1]$ with $\lambda(E) > 0$.

3. Let p and q be positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$, and let $g \in L_q (= L_q(\mathbf{R}, \mathcal{L}, \lambda))$. For $f \in L_p$ and real y define the function $T_y f$ by $(T_y f)(x) = f(x - y)$, and the function Lf by

$$(Lf)(y) = \int (T_y f)(x) g(x) d\lambda(x).$$

Show that L is a continuous linear operator from L_p to L_{∞} .

4. Let (X, ρ) and (Y, σ) be metric spaces and let $f : X \rightarrow Y$ be onto and continuous.
 - a. If X is separable is Y separable?
 - b. If X is complete is Y complete?
5. Let S be a linear subspace of a Hilbert space H with the property that the only element of H orthogonal to every element of S is the zero element. Prove that S is dense in H .
6. Let P be a polynomial and let f be a Lebesgue integrable real valued function defined on \mathbf{R} . The function g is defined for real t by

$$g(t) = \int P(\sin xt) f(x) d\lambda(x).$$

Prove that g is continuous.