

# ANALYSIS QUALIFYING EXAMINATION

Fall, 1996  
Saturday, August 31, 1996 9:00-12:00  
Room 268 Carver

Instructions:

- Write your social security number on every page you hand in. Do NOT write your name on any sheet you turn in.
- Work no more than 6 problems. No credit will be given for more than 6 problems and, if you turn in more than 6 problems, the problems on which you achieved the best performance will be discarded in the grading.
- Work each problem on a separate piece of paper and clearly indicate the part and the problem number of that part.
- To pass you must work at least two problems from Part I and three problems from Part II.

## PART I. Complex Analysis

1. Let  $f$  be an integrable, real valued function defined on  $[0, 1]$ . For  $z \in \mathfrak{C} - [0, 1]$ , where  $\mathfrak{C}$  is the set of complex numbers, define

$$F(z) = \int_0^1 \frac{f(x)}{x - z} dx.$$

- Show that  $F$  is well defined on  $\mathfrak{C} - [0, 1]$ .
- Show that  $F$  is analytic on  $\mathfrak{C} - [0, 1]$ .

2. Let  $D = \Delta[0, 1]$  denote the unit disk. Let  $f$  be analytic on  $D$  and let  $g$  be analytic and one-to-one on  $D$  with  $f(D) \subset g(D)$  and  $f(0) = g(0) = 1$ .

Show that

- $|f'(0)| \leq |g'(0)|$ .
- For each  $0 < r < 1$ ,

$$\max_{|z|=r} |f(z)| \leq \max_{|z|=r} |g(z)|.$$

3. Let  $f(z) = \frac{3z + 4 - 3i}{(z - 3i)(z + 2)}$ . Find a Laurent series that represents  $f$  in the annulus

$$A = \{z : 1 < |z + 1| < \sqrt{10}\}.$$

4. Prove the Casorati-Weierstrass Theorem without using the Picard theorem. That is, show that if  $z = \alpha$  is an essential singularity of the analytic function  $f$  and  $\gamma$  is any complex number, then in any neighborhood of  $\alpha$ ,  $f(z)$  comes arbitrarily close to  $\gamma$ .

## PART II. Real Analysis

1. Suppose that  $f$  is an absolutely continuous function on  $[0, 1]$  such that  $|f'| \leq M$  a.e..

- i. Show that  $|f(x) - f(y)| \leq M|x - y|$  for  $x, y \in [0, 1]$ .
- ii. Does the inequality of part (i) hold if the function  $f(\cdot)$  is only assumed to be of bounded variation? Support your answer with either a proof or a counter example.

2. Suppose  $f \in \mathcal{L}^1(0, 1)$  and  $\int_0^1 e^{2n\pi it} f(t) dt = 0$  for  $n = 0, \pm 1, \pm 2, \dots$ . Prove that  $f = 0$  a. e. .

3. Let  $\{a_n\}_{n=1}^\infty$  be a sequence of real numbers which is bounded above by some real number  $M$ . Some authors define  $\limsup_{n \rightarrow +\infty} a_n$  to be

$$\sup\{b \mid b = \lim_{k \rightarrow +\infty} a_{n_k} \text{ where } \{a_{n_k}\}_{k=1}^\infty \text{ is a subsequence of } \{a_n\}_{n=1}^\infty\}$$

while others define it to be  $\lim_{n \rightarrow \infty} b_n$  where we set  $b_n = \sup\{a_k \mid k = n, n+1, \dots\}$  for convenience. Carefully prove that these definitions are equivalent.

4. Let  $E$  be a measurable subset of  $\mathbb{R}^1$  and let  $f_n \rightarrow f$  a.e. where  $f \in \mathcal{L}^1(E)$ . Show that  $\int_E |f_n(x) - f(x)| dx \rightarrow 0$  as  $n \rightarrow \infty$  if and only if  $\int_E |f_n(x)| dx \rightarrow \int_E |f(x)| dx$  as  $n \rightarrow \infty$ .

5. Construct a function on  $[0, 1]$  which is monotone increasing and discontinuous precisely at the rationals. You must rigorously verify that the function you construct has the desired properties.

6. Let  $f(x) = \sin(x^\alpha)$  on  $(0, \infty)$  where  $\alpha \in \mathbb{R}$ , the real numbers. For what values of  $\alpha$  is  $f \in \mathcal{L}^1[0, \infty)$ ? For what values of  $\alpha$  does  $\int_0^\infty f(x) dx$  converge in the sense of Riemann? Support your answers with proofs.