

ANALYSIS QUALIFYING EXAMINATION

Fall, 2006

Saturday, August 19, 2006, 9:00 am - 12:00 noon

Room 305 Carver

Instructions:

- Write your university identification number on every page that you turn in. Do **NOT** write your name on any page that you turn in.
- Work no more than 6 problems. No credit will be awarded for more than 6 problems. If you turn in more than 6 problems, then all will be graded and your score will be the total of the scores on the 6 lowest scoring problems.
- Work each problem on a separate piece of paper and clearly indicate the problem and problem part on each page.
- To pass you must work on at least 2 problems from Part I and at least 3 problems from Part II, and receive substantial credits from both parts. In the grading, one completely correct solution will be counted as more than two "half correct" solutions.

Part I. Complex Analysis

1. Suppose that f is an entire function, and suppose there exists a constant C such that for every $|z| > 1$, $|f(z)| \leq Cz^2$. Prove that f is a polynomial of degree ≤ 2 .

2. Evaluate the integral

$$\int_0^{\infty} \frac{x^{1/2}}{1+x^4} dx.$$

3. Suppose f is analytic on $A = \{z : 0 < |z| < 1\}$ and $|f(z)| \leq \log \frac{1}{|z|}$ for all $z \in A$. Show that $f \equiv 0$.

4. Suppose $f : D \rightarrow \mathbf{C}$ is analytic and non-constant with $\operatorname{Re} f(z) \geq 0$ for all $z \in D$. (Here D is the open unit disk.)

(a) Show that $\operatorname{Re} f(z) > 0$ for all $z \in D$.

(b) Show that, if $f(0) = 1$, then

$$|f(z)| \leq \frac{1+|z|}{1-|z|}$$

for all $z \in D$.

Part II. Real Analysis

1. Suppose f is an integrable function on \mathbf{R} ; show that

$$\lim_{a \rightarrow 0} \int_{\mathbf{R}} |f(x) - f(x - a)| dx = 0.$$

2. Suppose μ and ν are finite Borel measures on \mathbf{R} such that $\mu \ll \nu$ and the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$ satisfies $0 < m < \frac{d\mu}{d\nu} < M$ for some constants m and M . Show that if $f \in L^2(\mathbf{R}, \nu)$, then $f \in L^2(\mathbf{R}, \mu)$, and that the inclusion map from $L^2(\mathbf{R}, \nu)$ to $L^2(\mathbf{R}, \mu)$ is one-to-one and onto.

3. Let $(X, \|\cdot\|)$ be a Banach space.

(a) Show that there is an inner product $\langle \cdot, \cdot \rangle$ on X such that $\langle x, x \rangle = \|x\|^2$ for all $x \in X$ if and only if

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

for all x and y in X .

(b) Use the result of the previous part to show that $L^1(0, 1)$ is not a Hilbert space.

4. Let $f : [0, 1] \rightarrow \mathbf{R}$ be a measurable function and $E \subseteq \{x : f'(x) \text{ exists}\}$. If $\mu(E) = 0$, show that $\mu(f(E)) = 0$.

5. Prove that $f \in L^2[0, 1]$ if and only if $f \in L^1[0, 1]$ and there exists a monotone increasing function g on $[0, 1]$ such that

$$\left| \int_a^b f(x) dx \right|^2 \leq (g(b) - g(a)) \cdot |b - a|$$

for all $0 \leq a \leq b \leq 1$.

6. Let (X, μ) be a σ -finite measurable space. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence in $L^2(X, \mu)$ such that $\|f_n - f_{n+1}\|_2 \leq 2^{-n}$ for all $n \geq 1$. Prove that there exists $f \in L^2(X, \mu)$ such that $f_n \xrightarrow{\text{a.e.}} f$ and $f_n \xrightarrow{\|\cdot\|_2} f$.