ANALYSIS QUALIFYING EXAM

Fall 2001
August 25, 2002, 9:00 am - 12:00 noon
Room 408 Carver

Instructions

- Write your complete social security number on every page that you turn in. Do NOT write your name on any sheet that you turn in.
- Work 6 problems, with at least 2 from Part I and at least 3 from Part II. No credit will be given for additional problems, and if additional problems are turned in, only the worst ones will be counted.
- Work each problem on a separate sheet of paper, and clearly indicate the part and problem number.
- To pass, you must receive substantial credit from each part. One correct problem will be counted more than two "half correct" problems in the grading.

Part I: Complex Analysis

1. Evaluate $\int_\Gamma \cot(z)\,dz$, where $\Gamma = \{z : |z - i| = 2\}$ with the positive orientation.

2. Let $G = \{z : |z + 1| > 1 \text{ and } |z + 2| < 2\}$. Find a function that maps G conformably onto the unit disk.

3. Evaluate the integral

$$\int_0^\infty \frac{dx}{x^{2/3}(1 + x)}$$

by a suitable complex contour integration.

4. Find the number of zeros of $f(z) = z^5 + z^2 + z + 4$ in the following regions:

(a) $\{z : 1 < |z| < 2\}$, and

(b) $\{z : \mathrm{Re}(z) > 0\}$.

Part II: Real Analysis

1. Let $\{F_n\}$ be a decreasing sequence of nonempty closed sets of real numbers:

$$F_0 \supset F_1 \supset F_2 \supset \cdots$$

Show that if one of the F_n is bounded, then $\bigcap_{i=0}^{\infty} F_i \neq \emptyset$.

Give a counterexample to show that the conclusion is not necessarily true if we do not require boundedness.

2. Let $f \in C^0([a,b])$ (that is, f is continuous on the interval $[a,b]$.) Suppose that f is differentiable in (a,b), and that $\lim_{x\to a^+} f'(x)$ exists. Is f differentiable at a?

Conversely, suppose $f \in C^0([a,b])$ and f is differentiable in $[a,b)$. Is it true that $\lim_{x\to a^+} f'(x)$ exists?

3. Evaluate the limits

(a)

$$\lim_{n\to\infty} \int_0^{\infty} e^{-x} \sin(\frac{x}{n})\, dx$$

(b)

$$\lim_{n\to\infty} \int_0^{\infty} e^{-x} \cos(\frac{x}{n})\, dx$$

4. Let μ be a positive measure on a measure space $(X, \mathcal{M})$, and let $\{f_n\}$ be a sequence of measurable functions $f_n : X \to \mathbb{R}$ which are non-negative, converge to a function f almost everywhere, and $f_n \leq f$ a.e. for all n. Prove that $\int f\, d\mu = \lim_n \int f_n\, d\mu$.

5. Recall a function f defined on an interval $[a,b]$ is *Lipschitz continuous* if there is a constant $M > 0$ such that for any $a \leq x \leq y \leq b$ we have

$$|f(y) - f(x)| \leq M|y - x|.$$

(a) Prove that if f is Lipschitz continuous on $[a,b]$, then f is absolutely continuous.

(b) Give an example of a function which is continuous but not absolutely continuous. Also give an example of a function which is absolutely continuous but not Lipschitz continuous.

6. Let (X, μ) be a σ-finite measure space. Let $1 \leq p \leq \infty$ and $g \in L^\infty$. Show that the operator $T : L^p \to L^p$, $Tf = gf$, is bounded, and $\|T\| = \|g\|_\infty$.