APPLIED MATH QUALIFYING EXAMINATION

Spring, 2009
Thursday, January 8, 9:00am-1:00pm
Room 305 Carver

Instructions

• Write your student ID number on every page that you turn in. Do NOT write your name on any page you turn in.
• Turn in solutions to 6 problems. No credit will be given for additional problems.
• Start each problem on a separate sheet of paper, with the problem number clearly stated at the top. SHOW ALL WORK.
• In the event that you believe a problem has a misprint or is improperly stated, ask the proctor for a clarification. Problems are not to be interpreted trivially.

Problems:

1. Consider the integro-differential boundary value problem

\[ -u'' + \int_0^1 xy u(y) \, dy = f(x), \quad 0 < x < 1, \quad u(0) = 0, \quad u'(1) = 0. \]

A function \( u(x) \) satisfies (1) if and only if \( u(x) \) and \( \alpha \in \mathbb{R} \) simultaneously satisfy

\[ u(x) = \int_0^1 g(x, s) [f(s) - \alpha s] \, ds, \quad \alpha = \int_0^1 y u(y) \, dy, \]

where \( g(x, s) \) is the Green's function for the operator \(-d^2/dx^2\) subject to the boundary conditions \( u(0) = 0, \quad u'(1) = 0\). Find \( g(x, s) \), the numerical value of \( \alpha \), and the explicit integral representation of \( u(x) \).

2. Let \( A \) and \( B \) be \((n \times n)\) matrices satisfying \( \| I - BA \| < 1 \), where \( \| \cdot \| \) is an operator norm on \( \mathbb{R}^n \). (In this case \( B \) is said to be an approximate inverse of \( A \).) Prove that both \( A \) and \( B \) are invertible and that

\[ \| A^{-1} - B \| \leq \frac{\| B \| \| I - BA \|}{1 - \| I - BA \|}. \]

Make your proof as complete as possible.

3. Define an operator \( A \) on \( L^2(-1, 1) \) by

\[ (Au)(x) = xu(x) + \theta \int_{-1}^1 u(x) \, dx, \]

where \( \theta \) is a positive constant. Show that \( A \) is bounded and self-adjoint. Describe the spectrum of \( A \) completely.
4. Does the equation \( T(\phi) = \sum_{n=1}^{\infty} n!\phi^{(n)}(n) \) define a distribution on \( \mathbb{R} \)? Justify your answer carefully.

5. Carry out the Gram-Schmidt procedure to find an orthonormal basis for the subspace \( M \) of \( L^2(-1,1) \) spanned by \( \{1, x, x^2, x^3\} \), and find the orthogonal projection of \( \sin(\pi x) \) onto \( M \).

6. The sequence of locally integrable functions \( \{f_k\}_{k=1}^{\infty} \) given by
\[
f_k(x) = k^2 H(x) e^{-kx}
\]
defines a sequence \( \{T_k\}_{k=1}^{\infty} \) of distributions on \( \mathbb{R} \). Find the limiting distribution \( T = \lim_{k \to \infty} T_k \) and verify that the limit exists in the sense of distributions.

7. Set \( U(x, t) = \frac{1}{2} H(t - |x|) \). Show that \( U \) is a fundamental solution with pole at \((0,0) \in \mathbb{R}^2 \) for the operator
\[
\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}.
\]

8. Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \), and define the functional \( J \) on \( H_0^1(\Omega) \setminus \{0\} \) by
\[
J(u) = \frac{\int_{\Omega} |\nabla u|^2 \, dx}{\int_{\Omega} |u|^2 \, dx}.
\]
Show that the critical points of \( J \) are exactly the eigenfunctions of the operator \( -\Delta \) subject to the homogeneous Dirichlet boundary condition \( (u = 0 \ on \ \partial \Omega) \).

9. Define the integral operator \( K \) on \( L^2(0,1) \) by
\[
Kf(x) = \int_0^1 \frac{f(y)}{\sqrt{|x-y|}} \, dy.
\]
Show that \( K \) is a compact operator. (Hint: Write \( K \) as the sum of a compact operator and a “small” operator.)