APPLIED MATH QUALIFYING EXAMINATION

Spring, 2009
Thursday, January 8, 9:00am-1:00pm
Room 305 Carver

Instructions

- **Write your student ID number on every page that you turn in**. Do **NOT** write your name on any page you turn in.
- Turn in solutions to 6 problems. No credit will be given for additional problems.
- Start each problem on a separate sheet of paper, with the problem number clearly stated at the top. **SHOW ALL WORK.**
- In the event that you believe a problem has a misprint or is improperly stated, ask the proctor for a clarification. Problems are not to be interpreted trivially.

Problems:

1. Consider the integro-differential boundary value problem

(1)
$$-u'' + \int_0^1 xyu(y)\,dy = f(x),\ 0 < x < 1, \quad u(0) = 0,\ u'(1) = 0.$$

   A function $u(x)$ satisfies (1) if and only if $u(x)$ and $\alpha \in \mathbb{R}$ simultaneously satisfy
$$u(x) = \int_0^1 g(x,s)[f(s) - \alpha s]\,ds, \qquad \alpha = \int_0^1 yu(y)\,dy,$$
   where $g(x,s)$ is the Green's function for the operator $-d^2/dx^2$ subject to the boundary conditions $u(0) = 0$, $u'(1) = 0$. Find $g(x,s)$, the numerical value of α, and the explicit integral representation of $u(x)$.

2. Let A and B be $(n \times n)$ matrices satisfying $\|I - BA\| < 1$, where $\|\cdot\|$ is an operator norm on $\mathbb{R}^n$. (In this case B is said to be an approximate inverse of A.) Prove that both A and B are invertible and that
$$\|A^{-1} - B\| \le \frac{\|B\|\,\|I - BA\|}{1 - \|I - BA\|}.$$
   Make your proof as complete as possible.

3. Define an operator A on $L^2(-1,1)$ by
$$(Au)(x) = xu(x) + \theta \int_{-1}^1 u(x)\,dx,$$
   where θ is a positive constant. Show that A is bounded and self-adjoint. Describe the spectrum of A completely.

4. Does the equation $T(\phi) = \sum_{n=1}^{\infty} n!\phi^{(n)}(n)$ define a distribution on $\mathbb{R}$? Justify your answer carefully.

5. Carry our the Gram-Schmidt procedure to find an orthonormal basis for the subspace M of $L^2(-1,1)$ spanned by $\{1, x, x^2, x^3\}$, and find the orthogonal projection of $\sin(\pi x)$ onto M.

6. The sequence of locally integrable functions $\{f_k\}_{k=1}^{\infty}$ given by
$$f_k(x) = k^2 H(x) e^{-kx}$$
defines a sequence $\{T_k\}_{k=1}^{\infty}$ of distributions on $\mathbb{R}$. Find the limiting distribution $T = \lim_{k\to\infty} T_k$ and verify that the limit exists in the sense of distributions.

7. Set $U(x,t) = \frac{1}{2}H(t - |x|)$. Show that U is a fundamental solution with pole at $(0,0) \in \mathbb{R}^2$ for the operator
$$\frac{\partial^2}{\partial^2 t} - \frac{\partial^2}{\partial^2 x}.$$

8. Let Ω be a bounded open subset of $\mathbb{R}^n$, and define the functional J on $H_0^1(\Omega) \setminus \{0\}$ by
$$J(u) = \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega |u|^2\,dx}.$$
Show that the critical points of J are exactly the eigenfunctions of the operator $-\Delta$ subject to the homogeneous Dirichlet boundary condition ($u = 0$ on $\partial\Omega$).

9. Define the integral operator K on $L^2(0,1)$ by
$$Kf(x) = \int_0^1 \frac{f(y)}{\sqrt{|x-y|}}\,dy.$$
Show that K is a compact operator. (Hint: Write K as the sum of a compact operator and a "small" operator.)