APPLIED MATH QUALIFYING EXAMINATION

Spring 2008
Thursday, January 10 9:00am-1:00pm
Room 305 Carver

Instructions:

- **Write your ISU ID number on every page that you turn in.** Do **NOT** write your name on any sheet you turn in.
- Turn in solutions to 6 problems. No credit will be given for additional problems.
- Start each problem on a separate sheet of paper, with the problem number clearly stated at the top. **SHOW ALL WORK**
- Every effort has been made to state the problems clearly and without misprints. However, in the event that you believe a problem is improperly stated, explain the problem to a proctor. Problems should not be interpreted in a way that they become trivial.

Problems:

1. Let $Lu = \beta u' + xu$ on the domain $D_L = \{u \in H^1(0,1) : u(0) + \alpha u(1) = 0\}$.

   a) For which $\alpha, \beta \in \mathbb{C} : \beta \neq 0$ is L self-adjoint?

   b) For which $\alpha \in \mathbb{R}, \beta \in \mathbb{R} \setminus \{0\}$ does there exist a unique solution to $Lu = f$ for all $f \in L^2(0,1)$. Provide solvability conditions for f in exceptional cases.

2. Let A be a bounded linear operator on a linear subspace D_A of Hilbert space H. Prove there is an extension $\tilde{A}$ of A defined on all of H for which $\|A\| = \|\tilde{A}\|$.

3. Use the method of characteristics to solve the following PDE:

$$\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = u, \qquad u(0,y) = \sin y, \quad y > 0.$$

Describe the region on which the solution is uniquely defined.

4. Let f be a continuous function on $\mathbb{R}$ and let $u(x,y) = e^x f(x-y)$. Show that $u_y + u_x = u$ in the sense of distributions on $\mathbb{R}^2$.

5. A sequence of bounded operators $T_n : H \to H$ (H is a Hilbert space) goes to zero *uniformly* if $\|T_n\| \to 0$, *strongly* if $\|T_n u\| \to 0$ for all $u \in H$, and *weakly* if $< T_n u, v > \to 0$ for all $u, v \in H$. Show uniform convergence to zero implies strong convergence to zero and this implies weak convergence to zero and give examples to show the reverse implications do not hold.

6. Prove the Riemann-Lebesgue lemma: If $f \in L^1(\mathbb{R}^n)$, then $\lim_{s\to\infty} \hat{f}(s) = 0$, where s is real. (You may assume that the space of finite linear combinations of characteristic functions of rectangles are dense in $L^1(\mathbb{R}^n)$.)

7. Let $\mathcal{V} = \{H^1(0,1) : u(0) = 0\}$ and define for $u \in \mathcal{V}$, $J(u) = \int_0^1 (u' - u)^2 \, dx$.
(a) Find the Euler-Lagrange equations for the minimization problem

$$J(u) \to \text{ min:} \qquad u \in \mathcal{V} : u(1) = 1.$$

(b) Find the solution to the minimization problem.

8. For any test function $f \in L^p(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$ we define the translate of f, $\tau_h f(x) = f(x - h)$.
a) State the definition of $\tau_h T$ if $T \in \mathcal{D}'(\mathbb{R}^n)$.
b) Show that $\lim_{h\to 0} \tau_h T = T$ in the sense of distributions, for any $T \in \mathcal{D}'(\mathbb{R}^n)$.

9. Define the operator

$$(Tu)(x) = \int_0^x y u(y) \, dy; \qquad u \in L^2(0,1).$$

(a) Show that T is a compact operator
(b) Find T^*, the adjoint of T
(c) Show that T has dense range
(d) Show that 0 is in the continuous spectrum of T