# APPLIED MATH QUALIFYING EXAMINATION

Spring 2007
Saturday, January 6 9:00am-12:00 noon
Room 305 Carver

- **Write your social security number on every page that you turn in.** Do **NOT** write your name on any sheet you turn in.
- Turn in solutions to 6 problems. No credit will be given for additional problems.
- Start each problem on a separate sheet of paper, with the problem number clearly stated at the top. **SHOW ALL WORK.**
- Every effort is made to proofread the exam as carefully as possible, but misprints may occur. If you believe that a problem has been stated incorrectly, check with the proctor and indicate your interpretation in the solution. Do not interpret the problem in such a way that it becomes trivial.

1. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1,\ x>0, y>0\}$. Find the solution of

$$\begin{aligned} \Delta u &= 0 & (x,y) &\in \Omega \\ u(x,y) &= 0 & x = 0 &\text{ or } y = 0 \\ u(x,y) &= 1 & x^2+y^2 &= 1 \end{aligned}$$

2. Let $f \in C^1(\mathbf{R}^2)$, $f \geq 0$, and $\int_{\mathbf{R}^2} f\,dx = 1$. Calculate $\lim_{k\to\infty} k^\alpha f(kx)$ in the sense of distributions for any $\alpha > 0$.

3. Let

$$f(x) = \begin{cases} \log^2 x & x > 0 \\ 0 & x < 0 \end{cases}$$

a) Show that $f$ defines a tempered distribution on $\mathbb{R}$.

b) Compute $f'$ in the sense of distributions.

4. Let $T$ denote the multiplication operator $Tu(x) = a(x)u(x)$ on $L^2(0,1)$, where

$$a(x) = \begin{cases} x & 0 \leq x < \frac{1}{2} \\ 1 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

Describe the spectrum of $T$, making sure to identify all of the parts the spectrum. Is $T$ compact?

5. Consider the nonlinear boundary value problem

$$-u'' + \lambda(\sin^2 u - \cos^2 x) = 0 \quad 0 < x < 1$$
$$u = 0 \quad u'(1) = 0$$

in which $0 < \lambda < .5$.

a) Use a fixed point argument to prove that there is a unique solution $u_\lambda \in C[0,1]$.

b) Show that $|u_\lambda(x)| \leq 1$ for all $x \in [0,1]$.

Hint: The Green's function for the BVP: $-u'' = f(x)$ on $[0,1]$, $u(0) = u'(1) = 0$, is $g(x,s) = \max(x,s)$.

6. Let $(c,d) \subset (a,b) \subset \mathbf{R}$ and let

$$M = \{u \in L^2(a,b) : u \equiv 0 \text{ a.e. on } (c,d)\}$$

Show that $M$ is a closed subspace of $L^2(a,b)$, find $M^\perp$ and find an explicit formula for the orthogonal projection onto $M$.

7. Let $E = \{u \in C^1([0,1]) : \int_0^1 u(x)^2\,dx = 1, u(1) = 0\}$ and $J(u) = \int_0^1 u'(x)^2\,dx$. Solve the optimization problem

$$\min_{u \in E} J(u)$$

Give both the minimum value and the function for which the minimum is achieved.

8. If $\sum_{n=-\infty}^{\infty} |n|^p |a_n|^2 < \infty$ for some $p > 1$, show that

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

is continuous on $\mathbb{R}$.

9. Consider the operator $A$ in $L^2(-1,1)$ defined by

$$(Au)(x) = -\frac{2}{3}u(x) + \int_{-1}^{1} y^2 u(y)\,dy.$$

Determine $A^*$, $\mathcal{N}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A^*)$, and $\mathcal{R}(A^*)$. Does the identity $\mathcal{R}(A) = \mathcal{N}(A^*)^\perp$ hold in this case?