1. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1, \ x > 0, y > 0\}$. Find the solution of
\[
\Delta u = 0 \quad (x, y) \in \Omega \\
u(x, y) = 0 \quad x = 0 \text{ or } y = 0 \\
u(x, y) = 1 \quad x^2 + y^2 = 1
\]

2. Let $f \in C^1(\mathbb{R}^2), f \geq 0$, and $\int_{\mathbb{R}^2} f \, dx = 1$. Calculate $\lim_{k \to \infty} k^\alpha f(kx)$ in the sense of distributions for any $\alpha > 0$.

3. Let
\[
f(x) = \begin{cases} 
\log^2 x & x > 0 \\
0 & x < 0 
\end{cases}
\]

a) Show that $f$ defines a tempered distribution on $\mathbb{R}$.
b) Compute $f'$ in the sense of distributions.

4. Let $T$ denote the multiplication operator $Tu(x) = a(x)u(x)$ on $L^2(0, 1)$, where
\[
a(x) = \begin{cases} 
x & 0 \leq x < \frac{1}{2} \\
1 & \frac{1}{2} \leq x \leq 1 
\end{cases}
\]

Describe the spectrum of $T$, making sure to identify all of the parts the spectrum. Is $T$ compact?
5. Consider the nonlinear boundary value problem
\[-u'' + \lambda (\sin^2 u - \cos^2 x) = 0 \quad 0 < x < 1 \]
\[u = 0 \quad u'(1) = 0\]
in which $0 < \lambda < .5$.

a) Use a fixed point argument to prove that there is a unique solution $u_\lambda \in C[0, 1]$.  
b) Show that $|u_\lambda(x)| \leq 1$ for all $x \in [0, 1]$.

Hint: The Green’s function for the BVP: $-u'' = f(x)$ on $[0, 1]$, $u(0) = u'(1) = 0$, is $g(x, s) = \max(x, s)$.

6. Let $(c, d) \subset (a, b) \subset \mathbb{R}$ and let

\[M = \{u \in L^2(a, b) : u \equiv 0 \text{ a.e. on } (c, d)\}\]

Show that $M$ is a closed subspace of $L^2(a, b)$, find $M^\perp$ and find an explicit formula for the orthogonal projection onto $M$.

7. Let $E = \{u \in C^1([0, 1]) : \int_0^1 u(x)^2 \, dx = 1, u(1) = 0\}$ and $J(u) = \int_0^1 u'(x)^2 \, dx$. Solve the optimization problem

\[\min_{u \in E} J(u)\]

Give both the minimum value and the function for which the minimum is achieved.

8. If $\sum_{n=-\infty}^{\infty} |n|^p |a_n|^2 < \infty$ for some $p > 1$, show that

\[f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}\]

is continuous on $\mathbb{R}$.

9. Consider the operator $A$ in $L^2(-1, 1)$ defined by

\[(Au)(x) = -\frac{2}{3} u(x) + \int_{-1}^{1} y^2 u(y) \, dy.\]

Determine $A^*, \mathcal{N}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A^*)$, and $\mathcal{R}(A^*)$. Does the identity $\mathcal{R}(A) = \mathcal{N}(A^*)^\perp$ hold in this case?