Qualifying Examination in Applied Mathematics

Spring 2004

• Write your student identification number on every page of the solutions you hand in. Do not write your name.

• Hand in a total of 6 problems. (If more are turned in then only the first 6 will be graded.)

1. Calculate $\Delta \ln(x^2 + y^2)$ in the sense of distributions on $\mathbb{R}^2$. Justify your work. You can use that $\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$ in polar coordinates.

2. A distribution T on $\mathbb{R}$ is *even* if $< T, \check{\phi} >=< T, \phi >$ for all test functions ϕ, where $\check{\phi}(x) = \phi(-x)$. Likewise T is *odd* if $< T, \check{\phi} >= - < T, \phi >$. Prove that the Fourier transform of an even tempered distribution is even.

For $k = 1, 2, 3, \ldots$ determine whether $\delta^{(k)}(x)$ even, odd or neither. (Justify your answer.)

3. Find the solution of the Cauchy problem

$$y u_y - x u_x = 2u, \qquad u(x, 1) = x.$$

4. Let $Lu = (x - 2)u'' + (1 - x)u' + u$ on $(0, 1)$.

a) Show that $\{x - 1, e^x\}$ is a fundamental set for L on this interval.

b) Compute the formal adjoint L^*.

c) Find the Green's function for

$$Lu = f \qquad u'(0) = 0 \quad u(1) = 0$$

5. Let $Tu(x) = \int_0^x u(y)\, dy$ in $L^2(0, 1)$. Find the adjoint operator T^*. Describe the spectrum of T.

6. Show, using a fixed point argument that there exists a unique solution $y \in C[0,1]$ to the following boundary value problem

$$-\frac{d^2y}{dx^2} + \sin y = \sin(x), \quad x \in (0,1), \quad y(0) = y'(1) = 0.$$

(To get started: show $g(x,\xi) = \min(x,\xi)$ is the Green's function for $-\dfrac{d^2}{dx^2}$ with the above boundary conditions.)

7. Let

$$J(u) = \int_0^1 (u''(x))^2 - u(x)\sin(\pi x)\,dx \qquad u \in H^2(0,1) : u(0) = u(1) = 0$$

Find the Euler-Lagrange equation for minimizing J. What is the minimum value of J? (Remember this may involve finding additional boundary conditions for u. You may assume that a critical point u of J is sufficiently smooth, so that any necessary integrations by parts are valid.)

8. Let T be a closed densely defined operator on a Hilbert space H and let $R_\lambda = (T - \lambda I)^{-1}$ be its resolvent operator. Show that $||R_\lambda|| \leq \dfrac{1}{\lambda}$ for all $\lambda > 0$ if and only if T is a negative operator, i.e. $< Tu, u > \leq 0$ for all $u \in D(T)$.

9. Let $\mathcal{H}$ denote the Hilbert space of functions f : $\displaystyle\int_{-\infty}^{\infty} \frac{|f(x)|^2}{1+x^2}\,dx < \infty$, equipped with the inner product $\displaystyle < u, v >= \int_{-\infty}^{\infty} \frac{u\bar{v}}{1+x^2}\,dx$. Let M be the subspace defined by $M = \text{span}\{1, \Theta(x)\}$ where Θ denotes the Heaviside function. Furthermore define the function $G(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$. Determine the distance from G to M. Explain the reasoning behind any calculation you make.