Qualifying Examination in Applied Mathematics

Fall 1998

• Write your student identification number on every page of the solutions you hand in. Do not write your name.

• Hand in a total of 6 problems. (If more are turned in then only the first 6 will be graded.)

1. Consider the integral equation

$$u(x) + f(x) = \lambda \int_0^1 xyu(y)\,dy \qquad 0 < x < 1$$

a) For what values of λ is there one and only one solution u for a given $f \in L^2(0,1)$? For such λ find the solution.

b) For the remaining values of λ, under what conditions on f does a solution exist? Find all solutions for such functions f.

2. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$, and let $f = f(x,y)$ be a given function defined on $\partial\Omega$. Find the Euler-Lagrange equation for the problem of minimizing

$$J(u) = \iint_\Omega \sqrt{1 + |\nabla u|^2}\,dydx$$

subject to the constraints that $u = f$ on $\partial\Omega$ and $\displaystyle\iint_\Omega u^2\,dydx = 1$.

3. Consider a function $v = v(x,t)$ defined implicitly by

$$v(x,t) = f(x - tv(x,t)) \qquad f \in C^1(R)$$

a) Show that v satisfies *Burger's equation*, $v_t + vv_x = 0$, at any point (x,t) for which $1 + tf'(x - tv(x,t)) \neq 0$.

b) Let v be a solution as above, and $S_\xi = \{(x,t) : x - tv(x,t) = \xi\}$ for a fixed constant ξ. Show that v is constant on S_ξ. Conclude that S_ξ is a straight line.

c) Suppose $f'(\xi) < 0$ for some ξ. Prove that v cannot be continuous in the upper half plane $\{(x,t) : t > 0\}$. (Hint: Find $\xi_1 < \xi < \xi_2$ so that the lines S_{ξ_1} and S_{ξ_2} intersect.)

4. Let

$$f(x) = \begin{cases} \log x & x > 0 \\ 0 & x < 0 \end{cases}$$

Show that $f \in \mathcal{D}'(\mathbb{R})$ and compute f' in the sense of distributions.

5. a) Evaluate the convolution $\delta' * H$ where δ is the Dirac delta function and H is the Heaviside function.

b) Prove or disprove: If $f, g \in L^1(\mathbb{R}^n)$ and $f * g \equiv 0$, then either f or g is zero.

6. Compute the Green's function for

$$u'' + 4u = f \quad 0 < x < 1 \qquad u(0) + u'(0) = 0 \quad u(1) = 0$$

7. Let T be a bounded linear operator on a Hilbert space H and $S = (I + T^*T)$.

a) Show that S is one to one and $||S^{-1}|| \leq 1$.

b) If T is compact, what can be said about the spectrum of S? Be as precise as possible.

8. Solve the boundary value problem for Laplace's equation in the annulus $A = \{x : 1 < |x| < 2\} \subset \mathbb{R}^2$:

$$\Delta u = 0 \quad x \in A$$

$$u(x) = 0 \quad |x| = 2$$

$$u(x) = \begin{cases} 0 & |x| = 1, \quad x_2 < 0 \\ 1 & |x| = 1, \quad x_2 > 0 \end{cases}$$

by separation of variables.

9. Let u be a continuous function and satisfy $u_{tt} - u_{xx} = 0$ in $\mathcal{D}'(\Omega)$, where Ω is an open subset of the (x, t) plane, and suppose that the closed rectangle with vertices $(x, t), (x, t+2h), (x \pm h, t+h)$ is contained in Ω.

a) Show that u must satisfy the four point property

$$u(x, t) + u(x, t + 2h) = u(x - h, t + h) + u(x + h, t + h)$$

b) Show that there can be no solution of the following Dirichlet problem in the circle $D = \{(x, t) : x^2 + t^2 < 1\}$:

$$u_{tt} - u_{xx} = 0 \quad (x, t) \in D \qquad u(x, t) = x^2 \quad (x, t) \in \partial D \qquad u \in C(\bar{D})$$