Instructions:

• Write your social security number on every page that you turn in. Do NOT write your name on any sheet you turn in.

• Turn in solutions to 6 problems. No credit will be given for additional problems.

• Start each problem on a separate sheet of paper, with the problem number clearly stated at the top. **SHOW ALL WORK**

Problems:

1. Let $J(u) = \int_{-1}^{0} 2(u')^2 \, dx + \int_{0}^{1} (u')^2 + u(x) \, dx : \quad u \in H^1(-1,1) : u(-1) = -u(1)$. Find the Euler-Lagrange condition for minimization of the functional $J$. Determine the function that minimizes $J$.

2. Let $h(x, y) = \begin{cases} 1 - (x - y) & \text{if } x > y, \\ 1 - (y - x) & \text{if } y > x. \end{cases}$ Define the integral operator

   $$Tu(x) = \int_{-1}^{1} h(x, y)u(y) \, dy$$

   on $L^2(-1,1)$. Find the spectrum of $T$. Be sure to carefully distinguish the different parts of the spectrum. (Suggestion: to find solutions of $Tu = \lambda u$ you may want to use an equivalent ODE problem.) What is $||T||$?

3. Define the Sobolev space

   $$H^k(\mathbb{R}^n) = \{ u \in L^2(\mathbb{R}^n) : D^\alpha u \in L^2(\mathbb{R}^n) \ \forall \ |\alpha| \leq k \}$$

   Show that if $k > \frac{n}{2}$ then $H^k(\mathbb{R}^n) \subset C(\mathbb{R}^n)$. 

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4. Let \( \{u_n\}_{n=1}^\infty \) be a complete orthonormal set in a separable Hilbert space \( H \). If \( \{v_n\}_{n=1}^\infty \) is another sequence in \( H \) with \( \sum_{n=1}^\infty \|u_n - v_n\|^2 < 1 \), show that \( \{v_n\}_{n=1}^\infty \) is a basis for \( H \). (Suggestion: Define \( T x = \sum_{n=1}^\infty (x, u_n)v_n \), then show that \( I - T \) is a contraction on \( H \). What does this tell you about \( T \)?)

5. Show carefully that

\[
T(\phi) = \lim_{\epsilon \to 0^+} \int_{|x|>\epsilon} \frac{\phi(x)}{x} \, dx
\]

defines a distribution in \( \mathcal{D}'(\mathbb{R}) \). Calculate \( T' \) in the sense of distributions.

6. Define a distribution \( \mu \) on \( \mathbb{R}^2 \) by

\[
\mu(\phi) = \iint_{x>t} \phi(x,t) \, dx \, dt + \int_{-\infty}^\infty \phi(t,-t) \, dt \quad \phi \in C_0^\infty(\mathbb{R}^2)
\]

Show that \( \mu \) satisfies the wave equation \( u_{tt} - u_{xx} = 0 \) in the sense of distributions on \( \mathbb{R}^2 \).

7. Solve the following boundary value problem using the Green’s function approach.

\[
(e^x y'(x))' = h(x) \quad 0 < x < 1; \quad y(0) = 0, \quad y'(1) = 0.
\]

Show that the solution depends continuously upon the data in the sense that \( \|y\|_2 \leq C\|h\|_2 \). (Give \( C \) explicitly.)

8. Consider the Neuman problem for the heat equation

\[
u_t = \Delta u, \quad x \in \Omega, \quad t > 0
\]

\[
u(x,0) = u_0(x) \quad x \in \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad x \in \partial \Omega, \quad t > 0
\]

(Here \( \Omega \) is a bounded subset of \( \mathbb{R}^N \) and \( u_0 \in L^2(\Omega) \).)

a) Find a formula for the solution \( u(x,t) \) by separation of variables.

b) Let \( \bar{u}_0 \) denote the constant function which is the mean value of \( u_0 \), that is

\[
\bar{u}_0 = \frac{1}{m(\Omega)} \int_{\Omega} u_0(x) \, dx
\]

where \( m(\Omega) \) is the Lebesgue measure of \( \Omega \). Show that \( u(\cdot,t) \to \bar{u}_0 \) as \( t \to +\infty \).

9. Let \( S_n \) denote the infinite strip \( \{(x, y) : |x| < n\} \) and \( \chi_n \) the characteristic function of \( S_n \). Let

\[
f_n(x,y) = y\chi_n(x,y).
\]

Find \( F_n \), the Fourier transform of \( f_n \) and show that \( \lim_{n \to \infty} F_n \) (in the sense of distributions) equals the Fourier transform of the function \( f(x,y) = y \).