

APPLIED MATH QUALIFYING EXAMINATION

Fall 2006
Saturday, August 19 9:00am-12:00 noon
Room 305 Carver

Instructions:

- Write your social security number on every page that you turn in. Do **NOT** write your name on any sheet you turn in.
- Turn in solutions to 6 problems. No credit will be given for additional problems.
- Start each problem on a separate sheet of paper, with the problem number clearly stated at the top. **SHOW ALL WORK**

Problems:

1. Let

$$J(u) = \int_{-1}^0 2(u')^2 dx + \int_0^1 (u')^2 + u(x) dx : u \in H^1(-1, 1) : u(-1) = -u(1).$$

Find the Euler-Lagrange condition for minimization of the functional J . Determine the function that minimizes J .

2. Let $h(x, y) = \begin{cases} 1 - (x - y) & \text{if } x > y, \\ 1 - (y - x) & \text{if } y > x. \end{cases}$ Define the integral operator

$$Tu(x) = \int_{-1}^1 h(x, y)u(y) dy$$

on $L^2(-1, 1)$. Find the spectrum of T . Be sure to carefully distinguish the different parts of the spectrum. (Suggestion: to find solutions of $Tu = \lambda u$ you may want to use an equivalent ODE problem.) What is $\|T\|$?

3. Define the Sobolev space

$$H^k(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : D^\alpha u \in L^2(\mathbb{R}^n) \forall |\alpha| \leq k\}$$

Show that if $k > \frac{n}{2}$ then $H^k(\mathbb{R}^n) \subset C(\mathbb{R}^n)$.

4. Let $\{u_n\}_{n=1}^\infty$ be a complete orthonormal set in a separable Hilbert space H . If $\{v_n\}_{n=1}^\infty$ is another sequence in H with $\sum_{n=1}^\infty \|u_n - v_n\|^2 < 1$, show that $\{v_n\}_{n=1}^\infty$ is a basis for H . (Suggestion: Define $Tx = \sum_{n=1}^\infty (x, u_n)v_n$, then show that $I - T$ is a contraction on H . What does this tell you about T ?)

5. Show carefully that

$$T(\phi) = \lim_{\epsilon \rightarrow 0^+} \int_{|x| > \epsilon} \frac{\phi(x)}{x} dx$$

defines a distribution in $\mathcal{D}'(\mathbb{R})$. Calculate T' in the sense of distributions.

6. Define a distribution μ on \mathbb{R}^2 by

$$\mu(\phi) = \iint_{x>t} \phi(x, t) dx dt + \int_{-\infty}^{\infty} \phi(t, -t) dt \quad \phi \in C_0^\infty(\mathbb{R}^2)$$

Show that μ satisfies the wave equation $u_{tt} - u_{xx} = 0$ in the sense of distributions on \mathbb{R}^2 .

7. Solve the following boundary value problem using the Green's function approach.

$$(e^x y'(x))' = h(x) \quad 0 < x < 1; \quad y(0) = 0, \quad y'(1) = 0.$$

Show that the solution depends continuously upon the data in the sense that $\|y\|_2 \leq C\{\|h\|_2\}$. (Give C explicitly.)

8. Consider the Neuman problem for the heat equation

$$u_t = \Delta u, \quad x \in \Omega, \quad t > 0$$

$$u(x, 0) = u_0(x) \quad x \in \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad x \in \partial\Omega, \quad t > 0$$

(Here Ω is a bounded subset of \mathbb{R}^N and $u_0 \in L^2(\Omega)$.)

a) Find a formula for the solution $u(x, t)$ by separation of variables.

b) Let \bar{u}_0 denote the constant function which is the mean value of u_0 , that is

$$\bar{u}_0 = \frac{1}{m(\Omega)} \int_{\Omega} u_0(x) dx$$

where $m(\Omega)$ is the Lebesgue measure of Ω . Show that $u(\cdot, t) \rightarrow \bar{u}_0$ as $t \rightarrow +\infty$.

9. Let S_n denote the infinite strip $\{(x, y) : |x| < n\}$ and χ_n the characteristic function of S_n . Let

$$f_n(x, y) = y\chi_n(x, y).$$

Find F_n , the Fourier transform of f_n and show that $\lim_{n \rightarrow \infty} F_n$ (in the sense of distributions) equals the Fourier transform of the function $f(x, y) = y$.