1. Let \( R = [-h, h] \times [-k, k] \) be a rectangle in \( \mathbb{R}^2 \) and let \( \chi_R \) be its characteristic function.
   a. What is \( \partial \chi_R / \partial x \)? What is the support of this distribution?
   b. What is the Fourier transform of \( \chi_R \)?
   c. What is the Fourier transform of the characteristic function of the complement of \( R \)?
   d. Is this transform a temperate function? Support your answer with a reason.

2. The causal fundamental solution \( E(x, t) \), with pole at \((x, t) = (0, 0)\), for the time-dependent diffusion equation in an absorbing medium satisfies
   \[
   \frac{\partial E}{\partial t} - \frac{\partial^2 E}{\partial x^2} + q^2 E = \delta(x, t),
   \]
   with \( E(x, t) = 0 \) for \((x, t) \in \mathbb{R} \times (-\infty, 0)\). Find \( E(x, t) \) by changing dependent variables \( E = \exp(-q^2 t) F \), keeping in mind that both \( E \) and \( F \) are distributions.

3. Let \( X \) be the linear space consisting of the continuous functions on \([0, 1]\). Verify that
   \[
   \| f \|_1 = \int_0^1 |f(x)| \, dx
   \]
   is a norm on \( X \). Show that \( X \), equipped with the norm \( \| \cdot \|_1 \), is not a Banach space, that is, the space is not complete.

4. Consider the boundary value problem
   \[
   \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = -\mu u, \quad x > 0
   \]
   \[
   u(0, t) = \int_0^\infty \beta(x) u(x, t) \, dx
   \]
   \[
   u(x, 0) = \phi(x)
   \]
   Use the method of characteristics to show that the solutions have the form
   \[
   u(x, t) = \begin{cases} 
   B(t - x) \exp[-\mu x] & \text{if } x < t, \\
   \phi(x - t) \exp[-\mu t] & \text{otherwise}
   \end{cases}
   \]
where \( B \) satisfies the integral equation
\[
B(t) = \int_0^t \beta(t - x) \exp[-\mu t + \mu x] B(x) dx + \exp(-\mu t) \int_0^\infty \beta(x + t) \phi(x) dx \quad t \geq 0
\]

5. Let \( T u(x) = \frac{u(x)}{x} \) when \( u \in L^2(0, 1) \).
   a. Find the largest domain \( D(T) \) for which \( T : D(T) \to L^2(0, 1) \).
   b. Show that \( T \) is densely defined, closed, self-adjoint and unbounded on this domain.

6. Let \( K(x, y) = 1 + xy \) and define the integral operator
\[
T u(x) = \int_0^1 K(x, y) u(y) dy.
\]
   a. Find the point spectrum of \( T, \sigma_p(T) \).
   b. Are there any other elements of the spectrum of \( T \)? Support your answer with reasons for credit.

7. Fill in the details of the following alternative approach to solving the Neumann problem
\[
-\Delta u = f \text{ for } x \in \Omega \text{ with } \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega
\]
where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \).
   a. Give a necessary condition on the right hand side for the problem to be solvable.
   b. Show that for any \( \epsilon > 0 \) there is exactly one solution \( u_\epsilon \) of
\[
-\Delta u + \epsilon u = f \quad x \in \Omega \text{ with } \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega.
\]
   c. Show that \( \int_{\Omega} f(x) dx = 0 \) implies \( \int_{\Omega} u_\epsilon(x) dx = 0 \).
   d. Show that there exists \( u \in H^1(\Omega) \) such that \( u_\epsilon \to u \) weakly in \( u \in H^1(\Omega) \) as \( \epsilon \to 0 \) and that \( u \) is a solution of the Neumann problem.

8. Find all complex numbers, \( a, b, c \) such that
\[
\int_0^{2\pi} |e^x - a - be^{ix} - ce^{-ix}|^2 dx
\]
is a minimum and give the value of the minimum in terms of an easily computable (via elementary calculus) expression. Be sure to justify your answer for credit.

9. Let \( \ell^2 \) be the Hilbert space of infinite sequences \( x = \{x_i\}_{i=1}^\infty \) of real numbers with the norm \( \|x\| = \left( \sum_{i=1}^\infty |x_i|^2 \right)^{1/2} \). The right shift operator \( S : \ell^2 \to \ell^2 \) is defined by
\[
S\{x_1, x_2, x_3, \ldots\} = \{0, x_1, x_2, \ldots\}.
\]
   a. For each \( x \in \ell^2 \) and \( t \in \mathbb{R} \), there is an element \( y(t) \in \ell^2 \) given by
\[
y(t) = \sum_{n=0}^\infty \frac{t^n S^n x}{n!}.
\]
   Find \( y(t) \) and verify the convergence of the infinite series.
   b. Show that \( y(t) \) satisfies
\[
\frac{dy}{dt} = Sy, \quad y(0) = x.
\]