APPLIED MATH QUALIFYING EXAMINATION

Fall 2004
Saturday, Aug. 21 9:00am-12:00 noon

Instructions:

- **Write your student identification number on every page that you turn in.** Do **NOT** write your name on any sheet you turn in.
- Turn in solutions to 6 problems. No credit will be given for additional problems.
- Start each problem on a separate sheet of paper, with the problem number clearly stated at the top. **SHOW ALL WORK**

Problems:

(1) (a) Show carefully by the definition that $\sum_{k=-\infty}^{\infty} \delta(x - 2k\pi)$ is a distribution on $\mathbf{R}$.

(b) Verify that

$$\frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikx} = \sum_{k=-\infty}^{\infty} \delta(x - 2k\pi)$$

in the sense of distributions.

(c) Let $\phi \in C_0^\infty(\mathbf{R})$. Verify the Poisson summation formula

$$\sum_{k=-\infty}^{\infty} \phi(k) = \sum_{k=-\infty}^{\infty} \hat{\phi}(2\pi k),$$

where $\hat{\phi}(\omega) = \int_{-\infty}^{\infty} \phi(x) e^{-ix\omega}\, dx$ is the Fourier transform of ϕ.

(2) Let $(c, d) \subset (a, b) \subset \mathbf{R}$ and let $M = \{u \in L^2(a, b) : u \equiv 0 \text{ a.e. on } (c, d)\}$ Show that M is a closed subspace of $L^2(a, b)$, find $M^\perp$ and find an explicit formula for the orthogonal projection onto M.

(3) Let A be a bounded linear operator on a Banach space X with $||A|| < 1$. Show that $I - A$ has a bounded inverse and that

$$\frac{1}{1 + ||A||} \leq ||(I - A)^{-1}|| \leq \frac{1}{1 - ||A||}.$$

(4) Solve by the method of characteristics for $u(x, y)$, and state where your solution is valid:

$$u_x - \frac{y^3}{2} u_y = u, \quad u(0, y) = f(y).$$

(5) Define the integral operator $K : L^2(0, 1) \to L^2(0, 1)$ by

$$(Ku)(x) = \int_0^1 (2 - 4x + 6y) u(y)\, dy.$$

Determine $\mathcal{R}(K)$, $\mathcal{N}(K)$ and all of the non-zero eigenvalues, and corresponding eigenfunctions, of K.

(6) Let $E = \{u \in C^1([0,1]) : \int_0^1 u(x)^2\,dx = 1, u(1) = 0\}$ and $J(u) = \int_0^1 u'(x)^2\,dx$. Solve the optimization problem

$$\min_{u\in E} J(u).$$

Give both the minimum value and the function for which the minimum is achieved.

(7) Assuming $\alpha > 0$, the solution of the boundary value problem

$$0 = u_{rr} + \tfrac{2}{r}u_r + g(r), \quad 0 < r < R,$$
$$u_r(0) = 0,\ u_r(R) + \alpha u(R) = 0.$$

can be written in the form

$$u(r) = \int_0^R G(s,r)g(s)s^2\,ds.$$

Verify this by finding $G(s,r)$.

(8) Let $[a,b] \subset \mathbf{R}$ be a bounded interval. A function $u(x)$ defined on $[a,b]$ is said to be Hölder continuous on $[a,b]$ with exponent $\alpha \in (0,1)$ if there is a constant C such that

(1) $$|u(x) - u(y)| \le C|x-y|^\alpha, \qquad \forall x,y \in [a,b].$$

Let $C^\alpha[a,b] = \{u \in C[a,b] : u \text{ satisfies (1)}\}$. If $u \in C^\alpha[a,b]$ then the number

$$[u]_\alpha = \sup\{\frac{|u(x)-u(y)|}{|x-y|^\alpha} : x,y \in [a,b],\ x \neq y\}$$

is well-defined. Show that i) $C^\alpha[a,b]$ is a Banach space with the norm

$$\|u\|_\alpha = \|u\|_\infty + [u]_\alpha,$$

and ii) $C^\alpha[a,b] \subset C[a,b]$ is a compact embedding.

(9) Define $H_0^1(0,1)$ in the usual way as the closure of $C_0^\infty(0,1)$ in the norm

$$\|u\|_1 = (\int_0^1 |u'(x)|^2\,dx)^{\frac{1}{2}}$$

(a) Show that $H_0^1(0,1)$ is continuously embedded in $C[0,1]$.
(b) Show that the distribution $\delta_a(\phi) = \phi(a)$, for $a \in (0,1)$, is a continuous linear functional on $H_0^1(0,1)$.
(c) According to the Riesz representation theorem there is a unique function $v \in H_0^1(0,1)$ such that $\delta_a(u) =< u,v >_1$ for all $u \in H_0^1(0,1)$, where $< u,v >_1$ is the corresponding inner product. Find v.