

Directions: Write each solution on a separate page. Submit solutions in same order as questions. Write the last four digits of your student ID (social security number) at the top of each page. Do not write your name on your paper.

All steps must be justified by computation or explanation. Greater weight will be given to one whole (correct) solution than to two error-free but incomplete solutions. To demonstrate adequate breadth, significant work must be done from each of Part I and Part II.

Part I

Note: Throughout Part I, \mathbb{Z}_n denotes the group or ring (as appropriate) of integers modulo n .

1. Let G be a simple, nonabelian group. Prove that G is isomorphic to a subgroup of $\text{Aut}(G)$.
2. Let $\alpha = (1\ 2\ 3\ 4)(5\ 6) \in A_6$. Find the order of the conjugacy class of α in A_6 .
3. Prove that every group of order 175 is isomorphic to either $\mathbb{Z}_7 \times \mathbb{Z}_{25}$ or $\mathbb{Z}_7 \times \mathbb{Z}_5 \times \mathbb{Z}_5$.
4. Let R and S be rings, and let I be an ideal of R and J and let J be an ideal of S . Prove that $I \times J$ is an ideal of $R \times S$ and that $(R \times S)/(I \times J) \cong R/I \times S/J$.
5. Classify each of the following rings as accurately as possible as a:
 - (i) field,
 - (ii) Euclidean domain,
 - (iii) principal ideal domain,
 - (iv) unique factorization domain,
 - (v) integral domain,
 - (vi) none of the above.(For example, \mathbb{Z} is a Euclidean domain but not a field.)
 - (a) \mathbb{Z}_9
 - (b) $\mathbb{R}[x, y]$
 - (c) $\mathbb{Z}[x]$
 - (d) $\mathbb{Z}_{11}[x]$
 - (e) $\mathbb{Q}[x]/(x^2 + 4)$.

(over)

Part II

6. Find the Jordan Canonical Form of $A = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

7. Let V and W be finite dimensional complex vector spaces and let $T : V \rightarrow W$ be a linear transformation with $\text{range}(T) = W$ and $\ker(T) = \{0\}$. Prove that there is a unique inverse linear transformation T^{-1} .

8. Let V be the (infinite dimensional) vector space of continuous real-valued functions on $[0, 1]$. Define an inner product on V by $(f, g) = \int_0^1 f(x)g(x) dx$. Define $T : V \rightarrow W$ by $T(f)(x) = xf(x)$ for all $x \in [0, 1]$. Prove each of the following.

- (a) T is a linear transformation.
- (b) T is self-adjoint.
- (c) T does not have any eigenvectors.

9. Let N be a normal, complex, $n \times n$ matrix. Prove that

$$\max_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \text{Re} \left(\frac{x^* N x}{x^* x} \right) = \max_{\lambda \in \sigma(N)} \text{Re}(\lambda).$$

($\text{Re}(c)$ is the real part of c and $\sigma(N)$ is the set of eigenvalues of N .)

10. For any $n \times n$ complex matrix A , define $\|A\| = \sum_{i,j=1}^n |a_{ij}|^2$. Prove that $\|A\|$ is invariant under unitary similarity, i.e., prove that $\|U^* A U\| = \|A\|$ for every unitary matrix U .