ALGEBRA QUALIFYING EXAM Spring 1997

Directions: Write each solution on a separate page. Please submit solutions in the same order as the questions. Write the last four digits of your student ID (social security number) at the top of each page. Do not write your name on the paper.

All steps must be justified by computation or explanation. Greater weight may be given to one whole (correct) solution than to several incomplete solutions. To demonstrate breadth, significant work must be done from each of Part I and Part II.

PART I

1. Let G be a group, and $A = Aut(G)$ be the group of automorphisms of G. Prove that if card $(G) \geq 3$, then card $(A) \geq 2$.

2. Let X be a finite set, and G a transitive permutation group on X.

   a) Let N be a normal subgroup of G, N not the identity subgroup. Prove there is no element of X which is fixed by every element of N.

   b) Let $A = \{(g, a) \in G \times X : g \cdot a = a\}$. Prove that G and A have the same number of elements.

3. An abelian group A has a subgroup B with $A/B \cong \mathbb{Z} \oplus \mathbb{Z}$. Prove that A is isomorphic to the group $B \oplus \mathbb{Z} \oplus \mathbb{Z}$.

4. Let R be a commutative ring with identity, and I be an ideal of R. Let

   $J := \{a \in R | a^n \in I$ for some positive integer n (which depends on a)$\}$.

   a) Show that J is an ideal of R.

   b) In the case $R = \mathbb{Z}$, let I be the principal ideal generated by 540. What is the corresponding J in part (a)?

5. Describe all maximal ideals in $\mathbb{Z}[x]$. Justify your answer fully.

PART II

6. Let $A \in M_6(\mathbb{R})$ with characteristic polynomial $p(x) = (x+2)^4(x-1)^2$ and minimal polynomial $m(x) = (x+2)^2(x-1)$. What are the possible Jordan canonical form(s) for A (up to permutation of Jordan blocks)?

7. Let $A \in M_n(\mathbb{C})$. Prove that A is invertible if and only if there is a polynomial $p \in C[x]$, with zero constant term, such that $p(A) = I$ (the $n \times n$ identity matrix).

8. Let $\mathcal{V} = M_n(\mathbb{C})$ be the vector space of complex $n \times n$ matrices equipped with the inner product
$$\langle A, B\rangle = tr(B^* A),$$
$A, B \in \mathcal{V}$. [You need not prove that this formula defines an inner product.] For $A \in M_n(\mathbb{C})$, let $L_A \in \mathcal{L}(\mathcal{V})$ be the linear transformation $L_A(B) = AB,\ B \in \mathcal{V}$. Prove

   i) If $A \in M_n(\mathbb{C})$ is a Hermitian matrix, then L_A is Hermitian.

   ii) If $A \in M_n(\mathbb{C})$ is a unitary matrix, then L_A is unitary.

9. Let $\mathcal{V}$ be a finite dimensional inner product space and $y_0 \in \mathcal{V}$, y_0 of unit length. Consider the mapping $T : \mathcal{V} \to \mathcal{V},\ Tx = x - 2\langle x, y_0\rangle y_0$.

   a) Prove that T is a linear transformation.

   b) Calculate the determinant of T.

10. Let $A, B \in M_2(\mathbb{R})$, both invertible, satisfying $B^{-1}AB = A^2$. Given that 1 is not an eigenvalue of A,

    i) Determine the eigenvalues of A.

    ii) Find $A, B \in M_2(\mathbb{R})$ satisfying the above conditions.