Algebra Qualifying Exam Spring 1996

Directions: *Write each solution on a separate page. Please submit solutions in the same order as the questions. Write the last four digits of your student ID (social security number) at the top of each page. Do not write your name on the paper.*
All steps must be justified by computation or explanation. Greater weight may be given to one whole (correct) solution than to several incomplete solutions. To demonstrate adequate breadth, significant work must be done from each of Part I and Part II.

Part I

1. Let π be a permutation of $\{1, 2, \ldots, n\}$ having cycle type $1^{e_1}2^{e_2}\ldots n^{e_n}$. In other words, decomposing π as a product of disjoint cycles, there are e_2 cycles of length $2, \ldots, e_n$ cycles of length n. How many elements of the symmetric group S_n commute with π?

2. Let σ and τ be distinct transpositions in the symmetric group S_3. Let G be a group, and let $f : \{\sigma, \tau\} \to G$ be a map with $f(\sigma) = x$ and $f(\tau) = y$. Prove that f extends to a homomorphism $S_3 \to G$ if and only if $x^2 = y^2 = (xy)^3 = 1$.

3. Let G be the group of invertible 2×2 matrices over the ring $\mathbb{Z}/3\mathbb{Z}$.
   a) What is the cardinality of G?

   b) How many conjugacy classes does G have?
   [Hint: Consider the invariant factors of the matrices in G.]

4. Let p be a prime. Let R be the ring of functions $f : (\mathbb{Z}/p\mathbb{Z}) \to (\mathbb{Z}/p\mathbb{Z})$, with $(f + g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = f(x)g(x)$ for x in $\mathbb{Z}/p\mathbb{Z}$. Let P be the subset of R consisting of polynomial functions.
   a) Prove that P is a subring of R.

   b) Prove that P is isomorphic to the ring $(\mathbb{Z}/p\mathbb{Z})^p$.

5. Let $e : D \to F$ be the embedding of an integral domain D into its field of fractions F. Suppose that $f : F \to R$ and $g : F \to R$ are (unital) ring homomorphisms such that $f \circ e(x) = g \circ e(x)$ for all x in D. Show that $f = g$.

Part II

6. Let A be a square complex matrix with $A^3 = A$. Prove that A is diagonalizable.

7. Let V be a finite-dimensional vector space over a field F. Let $T : V \to V$ be an endomorphism of V, with minimal polynomial $m(X)$. Let W be a subspace of V invariant with respect to T, and let $T_1 : W \to W$ be the restriction of T. Let $m_1(X)$ be the minimal polynomial of T_1.

   a) Show that there is a well-defined endomorphism $T_2 : V/W \to V/W$ of V/W given by $T_2(v + W) = Tv + W$.

   b) Let $m_2(X)$ be the minimal polynomial of T_2. Show that $m_1(X)$ divides $m(X)$ and $m(X)/m_1(X)$ divides $m_2(X)$.

   c) Prove, or give a counterexample to, the following statement:

$m_2(X) = m(X)/m_1(X)$.

8. Let V be a finite-dimensional vector space over a field F. Let μ be a non-zero element of F such that $\mu^n \neq 1$ for all positive integers n. Show that there is no pair S, T of automorphisms of V satisfying $ST = \mu TS$.

9. Let A be a 2×2 complex matrix with characteristic polynomial $x^2 - 2x - 3$. Give formulas for numbers a_n and b_n, in terms of n, such that $A^n = a_n A + b_n I$ for all positive integral n.

10. Show that an $n \times n$ complex matrix A is normal if and only if there exists an $n \times n$ unitary matrix V such that $A^* = AV$.