Algebra Qualifying Exam

January 10, 2009

Directions: Write the problem number and the last four digits of your student ID number at the top of each page. Do not write your name on your paper. Write each solution on a separate page. Submit solutions in the same order as the questions. All the steps must be justified by computation or explanation. Greater weight will be given to one whole (correct) solution than to two error-free but incomplete solutions. To demonstrate adequate breadth, significant work must be done from each of Part I and Part II.

Every effort is made to proofread the exam, but misprints may occur. If you believe that a problem has been stated incorrectly, check with the proctor and indicate your interpretation in the solution. Do not interpret the problem in a way that it becomes trivial.

Part I

1. Show that if a finite group G has a subgroup H of index n, then H contains a normal subgroup of G of index a divisor of $n!$.

2. For each integer $n > 1$, show that there is a nonabelian group of order n^3.

3. Let $SL_2(\mathbf{F}_3)$ be the special linear group over the finite field $\mathbf{F}_3$. Show that any Sylow 2-group of $SL_2(\mathbf{F}_3)$ is isomorphic to the quaternion group of order 8.

4. Suppose that each element x of a nonunital ring R satisfies $x^2 = x$. Show that R is commutative.

5. Let R be a commutative ring with identity, and I be an ideal of R. Let

$$J = \{a \in R \mid \exists\, n > 0 \text{ such that } a^n \in I\}$$

   (a) Show that J is an ideal of R.

   (b) In the case when R is the ring of integers, let I be the principal ideal generated by 360 and J be as defined above. Find $r \in R$ such that $J = \langle\, r \,\rangle$, the ideal generated by r.

(Part II is on next page.)

Part II

6. Let T be a linear operator on a finite dimensional vector space V over $\mathbb{C}$. If $T^p = \mathrm{id}_V$ for some prime number p and $\operatorname{Tr} T = 0$, show that $p \big| \dim V$.

   (**Hint:** You can use the fact that for a prime number p, the minimal polynomial of $e^{\frac{2\pi i}{p}}$ over $\mathbb{Q}$ is $1 + x + \cdots + x^{p-1}$.)

7. Find a single 2×2 unitary matrix $U \in M_2(\mathbb{C})$ such that, for all real numbers θ, the matrix

$$U^* \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} U$$

   is diagonal.

8. Let A be a 3×3 complex matrix. Show that A is invertible if and only if

$$I = c_1 A + c_2 A^2 + c_3 A^3$$

   for suitable scalars $c_1, c_2, c_3 \in \mathbb{C}$.

9. Let $A,\ B \in M_n(\mathbb{C})$. Define $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^* \mathbf{x}$ for $\mathbf{x}, \mathbf{y}$ in $\mathbb{C}^n$. Prove that the following conditions are equivalent:

   (a) $\langle A\mathbf{x}, \mathbf{y} \rangle = \langle B\mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x},\ \mathbf{y} \in \mathbb{C}^n$.

   (b) $\langle A\mathbf{x}, \mathbf{x} \rangle = \langle B\mathbf{x}, \mathbf{x} \rangle$ for all $\mathbf{x} \in \mathbb{C}^n$.

10. Let A be an $n \times n$ Hermitian matrix. For $1 \le k \le n$, let A_k be the $k \times k$ leading principal submatrix of A. Prove that A is positive definite if and only if $\det A_k > 0$ for all $k = 1, \ldots, n$.