Part I

1. Show that if a finite group $G$ has a subgroup $H$ of index $n$, then $H$ contains a normal subgroup of $G$ of index a divisor of $n!$.

2. For each integer $n > 1$, show that there is a nonabelian group of order $n^3$.

3. Let $SL_2(F_3)$ be the special linear group over the finite field $F_3$. Show that any Sylow 2-group of $SL_2(F_3)$ is isomorphic to the quaternion group of order 8.

4. Suppose that each element $x$ of a nonunital ring $R$ satisfies $x^2 = x$. Show that $R$ is commutative.

5. Let $R$ be a commutative ring with identity, and $I$ be an ideal of $R$. Let

$$J = \{a \in R \mid \exists n > 0 \text{ such that } a^n \in I\}$$

(a) Show that $J$ is an ideal of $R$.

(b) In the case when $R$ is the ring of integers, let $I$ be the principal ideal generated by 360 and $J$ be as defined above. Find $r \in R$ such that $J = \langle r \rangle$, the ideal generated by $r$.

(Part II is on next page.)
Part II

6. Let $T$ be a linear operator on a finite dimensional vector space $V$ over $\mathbb{C}$. If $T^p = \text{id}_V$ for some prime number $p$ and $\text{Tr} T = 0$, show that $p | \dim V$.

(\textbf{Hint:} You can use the fact that for a prime number $p$, the minimal polynomial of $e^{\frac{2\pi i}{p}}$ over $\mathbb{Q}$ is $1 + x + \cdots + x^{p-1}$.)

7. Find a single $2 \times 2$ unitary matrix $U \in M_2(\mathbb{C})$ such that, for all real numbers $\theta$, the matrix

$$
U^* \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix} U
$$

is diagonal.

8. Let $A$ be a $3 \times 3$ complex matrix. Show that $A$ is invertible if and only if

$$I = c_1 A + c_2 A^2 + c_3 A^3$$

for suitable scalars $c_1, c_2, c_3 \in \mathbb{C}$.

9. Let $A, B \in M_n(\mathbb{C})$. Define $\langle x, y \rangle = y^* x$ for $x, y$ in $\mathbb{C}^n$. Prove that the following conditions are equivalent:

(a) $\langle Ax, y \rangle = \langle Bx, y \rangle$ for all $x, y \in \mathbb{C}^n$.

(b) $\langle Ax, x \rangle = \langle Bx, x \rangle$ for all $x \in \mathbb{C}^n$.

10. Let $A$ be an $n \times n$ Hermitian matrix. For $1 \leq k \leq n$, let $A_k$ be the $k \times k$ leading principal submatrix of $A$. Prove that $A$ is positive definite if and only if $\det A_k > 0$ for all $k = 1, \ldots, n$. 

2