Directions: Write the problem number and the last four digits of your student ID number at the top of each page. Do not write your name on your paper. Write each solution on a separate page. Submit solutions in the same order as the questions. All the steps must be justified by computation or explanation. Greater weight will be given to one whole (correct) solution than to two error-free but incomplete solutions. To demonstrate adequate breadth, significant work must be done from each of Part I and Part II.

Part I

(1) Let $G$ be a finite group with $|G| = mp$ where $p$ is a prime and $m$ is a positive integer such that $2 \leq m < p$. Prove that $G$ is not simple.

(2) Find all finite groups with exactly two conjugacy classes.

(3) Prove that $A_5$ has exactly 5 subgroups of order 4.

(4) Let $p$ be a prime, $G$ a non-abelian group of order $p^3$, and $Z(G)$ the center of $G$. Show that $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

(5) Let $R$ be an integral domain, $a, b \in R$. Prove that $a$ and $b$ are associates if and only if they generate the same ideal of $R$.

Part II

(6) Let $A = \begin{bmatrix} -11 & 30 & 78 \\ -10 & 26 & 65 \\ 2 & -5 & -12 \end{bmatrix}$. Find the rational canonical form (invariant factors version) of $A$.

(7) Let $H, P \in \mathbb{C}^{n \times n}$ with $H$ Hermitian and $P$ positive definite. Prove that the largest eigenvalue of $H + P$ is greater than the largest eigenvalue of $H$.

(8) Let $H = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$ and define an inner product $\langle \cdot, \cdot \rangle_H$ on $\mathbb{C}^2$ by $\langle x, y \rangle_H = y^* H x$. Find an orthonormal basis (with respect to this inner product) for $\mathbb{C}^2$. (You may assume $\langle \cdot, \cdot \rangle_H$ is an inner product.)

(9) Let $V$ be a (possibly infinite dimensional) vector space over the field $F$, and $W$ a subspace of $V$. Prove that there exists a subspace $U$ of $V$ such that $V = U \oplus W$.

(10) Let $V$ be a finite dimensional inner product space, $T$ a linear operator on $V$, and $T^*$ the adjoint operator of $T$. Prove that $\text{im}(T^*) = \ker(T)^\perp$. 