ALGEBRA QUALIFYING EXAMINATION
JANUARY 2006

Directions: Write the problem number and the last four digits of your student ID number at the top of each page. Do not write your name on your paper. Write each solution on a separate page. Submit solutions in the same order as the questions. All the steps must be justified by computation or explanation. Greater weight will be given to one whole (correct) solution than to two error-free but incomplete solutions. To demonstrate adequate breadth, significant work must be done from each of Part I and Part II.

Part I

(1) Prove that every group of order 105 has a subgroup of order 21.

(2) Describe a method that allows construction of a finite field of order p^k where p is prime. Explain why the method works and use it to explicitly construct the addition and multiplication tables for a field of order 4.

(3) (a) Define the center of a ring.
(b) Prove or give a counter-example to the following statement:
The center of a ring R is an ideal of R.

(4) Let C be a cyclic normal subgroup of a finite group G, and let H be a subgroup of C. Prove that H is a normal subgroup of G.

(5) Let H and K be finite index subgroups of a (possibly infinite) group G. Prove that $H \cap K$ is a finite index subgroup of G.

Part II

The following fact may be assumed: For any positive definite matrix M there exists a positive definite matrix S such that $M = S^2$.

(6) Find the Jordan Canonical Form of $A = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

(7) Let F be a field and let $A \in F^{n \times n}$. Prove that A is invertible if and only if $I \in \text{Span}(A, A^2, \ldots, A^n)$

(8) Let A and B be $n \times n$ Hermitian matrices, and let A be positive definite. Show that for any $x \in \mathbb{C}^n$,

$$\lambda_{min}(A^{-1}B) \le \frac{x^*Bx}{x^*Ax} \le \lambda_{max}(A^{-1}B).$$

(9) Let H be a fixed positive definite matrix in $\mathbb{C}^{n \times n}$. Define a product

$$\langle\ ,\ \rangle_H : \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n} \to \mathbb{C} \quad \text{by} \quad \langle A, B \rangle_H = \text{trace}(B^*HA).$$

Prove $\langle\ ,\ \rangle_H$ is an inner product on the complex vector space $\mathbb{C}^{n \times n}$.

(10) For what values of $\gamma \in \mathbb{C}$ do there exist nonsingular $A, B \in \mathbb{C}^{n \times n}$ such that $AB = \gamma BA$?