Qualifying Exam **Algebra** **Spring 2004**

Directions: Write each solution on a separate page. Submit solutions in the same order as questions. Write the last four digits of your student ID number at the top of each page. Do not write your name on your paper.

All steps must be justified by computation or explanation. Greater weight will be given to one whole (correct) solution than to two error-free but incomplete solutions. To demonstrate adequate breadth, significant work must be done from each of Part I and Part II.

Part I

1. Let $(\mathbb{R}, +)$ be the real numbers under addition and let $+\infty$ and $-\infty$ be two additional symbols. Prove that it is impossible to define an operation $+$ on the set $G = \mathbb{R} \cup \{+\infty, -\infty\}$ so that $(G, +)$ is a group which contains $(\mathbb{R}, +)$ as an additive subgroup.

2. Suppose the G is a finite group acting transitively on a set X with element x. Show that the number of orbits of the stabiliser G_x of x acting on X is equal to the number of orbits in the diagonal action of G on $X \times X$.

3. Let p be a prime number, and let G be a group of order p^2. Show that G is abelian.

4. Let R be the free associative ring on two generators. Prove that for any n, R contains a subring isomorphic to the free associative ring on n generators.

5. Prove the following statement:

$$\exists f : \mathbb{N} \to \mathbb{Q}, \forall p(X) \in \mathbb{Q}[X], \exists n \in \mathbb{N}, f(n) \neq p(n).$$

Part II

6. Let $F[[x]]$ be the ring of formal power series.

(1) If $a = \sum_{i=0}^{\infty} a_i x^i$ is an element of $F[[x]]$, show that a is invertible if and only if $a_0 \neq 0$.
(2) Prove that $F[[x]]$ has exactly one prime element.

7. Let V be a 3-dimensional vector space over the field $\mathbb{Q}$ of rational numbers. Show that there is no automorphism θ of V such that $\theta^{-1} = 2\theta$.

8. (1) Define what is meant by a *basis* of a vector space.
(2) Show that each vector space has a basis.

9. Prove that the rank of a square complex matrix is not less than the cardinality of its multiset of non-zero eigenvalues.

10. Let $F \subseteq K \subseteq L$ be fields, where F a subfield of K and K is a subfield of L. If the dimension of K over F is k and the dimension of L over K is l, prove that the dimension of L over F is kl.