# Algebra Qualifying Exam
Spring 2003

**Directions:** Write each solution on a separate page. Submit solutions in same order as questions. Write the last four digits of your student ID (social security number) at the top of each page. Do not write your name on your paper.

All steps must be be justified by computation or explanation. Greater weight will be given to one whole (correct) solution than to two error-free but incomplete solutions. To demonstrate adequate breadth, significant work must be done from each of Part I and Part II.

## Part I

1. Prove that for every integer $n$, $n^7 - n$ is divisible by 42.

2. Prove that no group of order 1500 is simple.

3. Let $G$ be a group with trivial center. Prove that $\mathrm{Aut}(G)$ has trivial center.

4. Let $R$ be a ring and suppose that for every $r \in R$, $r^2 = r$. Prove that $R$ is commutative.

5. Let $I$ be the principal ideal of the ring $\mathbb{Z}[x]$ generated by $(x-1)$. Prove that $I$ is a prime ideal that is not maximal.

## Part II

6. Find all real square matrices that are both symmetric and nilpotent. Justify your claim to have found all such matrices.

7. Find, up to similarity, all complex square matrices with characteristic polynomial $x^4 - 2x^2 + 1$. Justify your answer.

8. Let $V$ be a finite dimensional vector space over a field $F$. Recall that the dual of $V$ is the vector space

$$V^* = \{\, f\colon V \to F \mid f \text{ is a linear transformation} \,\}.$$

Given a basis $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ for $V$, define $f_i \in V^*$ for $i = 1, \ldots, n$ by

$$f_i(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Prove that $\mathcal{B}^* = \{f_1, f_2, \ldots, f_n\}$ is a basis for $V^*$. Identify the point at which your proof fails if $V$ is not finite dimensional.

9. Let $A$ be a square complex matrix and suppose that $A$ is similar to $A^2$.
   (a) Let $\lambda$ be an eigenvalue of $A$. Prove that $|\lambda| \in \{0, 1\}$.
   (b) Prove that if $|\lambda| = 0$, its geometric and algebraic multiplicities must be equal.
   (c) Give an example to show that when $|\lambda| = 1$, the geometric and algebraic multiplicities need not be equal.

10. Let $\mathrm{Gl}(n, \mathbb{C})$ denote the group of all $n \times n$ nonsingular complex matrices (under matrix multiplication). Let $H$ be a finite group, and $f\colon H \to \mathrm{Gl}(n, \mathbb{C})$ a group homomorphism. Prove that for any $h \in H$, $f(h)$ is a diagonalizable matrix.