Qualifying Exam **Algebra** **Fall 2003**

Directions: Write each solution on a separate page. Submit solutions in the same order as questions. Write the last four digits of your student ID number at the top of each page. Do not write your name on your paper.

All steps must be justified by computation or explanation. Greater weight will be given to one whole (correct) solution than to two error-free but incomplete solutions. To demonstrate adequate breadth, significant work must be done from each of Part I and Part II.

Part I: Groups, Rings, etc.

1. For each pair of groups G, H below, give a reason why G and H are not isomorphic

   (1) $G = S_3$, $H = \mathbb{Z}_6$.
   (2) $G = \mathbb{Z}_{12} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$, $H = \mathbb{Z}_6 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_4$
   (3) $G = D_8$, $H = Q_8$.
   (4) $G = A_5$, $H = A_4 \oplus \mathbb{Z}_5$.

2. Show that any group of order 992 is not simple.

3. Show that the 2×2 matrix ring over the ring $\mathbb{Z}/5\mathbb{Z}$ of residues modulo 5 has no proper non-trivial ideals.

4. Find the smallest positive integer n for which there is an even permutation that does not lie in any Sylow subgroup of the alternating group A_n. Justify your answer.

5. Consider the set of all non-zero matrices of the form $\begin{bmatrix} a & 3b \\ b & a \end{bmatrix}$ in the 2×2 matrix ring over $\mathbb{Z}/5\mathbb{Z}$. Prove that this set forms a cyclic group under matrix multiplication.

Part II: Linear Algebra

6. Prove that each upper triangular matrix is similar to a lower triangular matrix.

7. State the Jordan Canonical Form Theorem. Use it to prove that if an $n \times n$ complex matrix A satisfies $A^2 = A$, then A is diagonalisable.

8. Find a single 2×2 unitary matrix $U \in M_2(\mathbb{C})$ such that, for all real numbers θ, the matrix

$$U^* \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} U$$

is diagonal.

9. Suppose that an $n \times n$ complex matrix $A \in M_n(\mathbb{C})$ commutes with both A^*A and AA^*. Prove that the column null space of A is the column null space of A^*. (The column null space of A is $\ker A$.)

10. Let $A \in M_n(\mathbb{Q})$. Use the following table of ranks of $(A - \lambda I)^s$ to find

(1) a Jordan matrix (in $M_n(\mathbb{C})$) similar to A, i.e., $JCF(A)$.
(2) the invariant factors of A, the minimal polynomial of A, and the characteristic polynomial of A. (Polynomials should be given in factored form).

	$s=0$	$s=1$	$s=2$	$s=3$
$\lambda = 2$	13	10	8	8
$\lambda = 1+i$	13	10	9	9
$\lambda = 1-i$	13	10	9	9