Algebra Qualifying Exam
Fall 2001

Directions: Write each solution on a separate page. Submit solutions in same order as questions. Write the last four digits of your student ID (social security number) at the top of each page. Do not write your name on your paper.

All steps must be be justified by computation or explanation. Greater weight will be given to one whole (correct) solution than to two error-free but incomplete solutions. To demonstrate adequate breadth, significant work must be done from each of Part I and Part II.

Part I

1. Let G be a finite group of order n, $a \in G$, and let k be an integer relatively prime to n. Prove that the order of a^k is equal to the order of a.

2. Let $\sigma = (1\,2\,3\,4)(5\,6\,7) \in S_7$. Show that the centralizer of σ is equal to $\langle\sigma\rangle$ (the cyclic subgroup generated by σ).

3. Let p be a prime and G a nonabelian group of order p^3.
   (a) Prove that $|Z(G)| = p$. ($Z(G)$ is the center of G.)
   (b) Let H be a subgroup of G of order p^2. Prove that $Z(G) < H$.

4. Let R be a principal ideal domain and $a_i \in R$, for $i = 1, 2, 3, \ldots$ Prove that, if $(a_1) \subseteq (a_2) \subseteq (a_3) \subseteq \cdots$, then, for some n, we have $(a_n) = (a_{n+1}) = (a_{n+2}) = \cdots$.

5. Prove or disprove: $\mathbb{Z}_5[x]/(x^2 + x + 2) \cong \mathbb{Z}_5 \times \mathbb{Z}_5$ (as rings).

Part II

6. Let B be a real symmetric $n \times n$ matrix such that, for all $x \in \mathbb{R}^n$, $x^T Bx > 0$. Prove that $z^* Bz > 0$ for every $z \in \mathbb{C}^n$, $z \neq 0$.

7. Let $A \in M_n(\mathbb{Q})$ satisfy $A^2 = 2I$. Prove that n must be even.

8. Let V be the vector space $\mathbb{C}^2$ with inner product
$$\left\langle \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\rangle = 4z_1\bar{w}_1 + 2z_1\bar{w}_2 + 2z_2\bar{w}_1 + z_2\bar{w}_2.$$
   Find a unitary linear transformation from V to V that is not a scaler multiple of the identity.

9. Let F be a field, V a finite dimensional vector space over F, and W a subspace of V. Let f be a linear functional on W (i.e., $f\colon W \to F$ is a linear transformation). Prove that there is a linear functional g on V such that, for all $w \in W$, $g(w) = f(w)$.

10. Suppose A and B are 6×6 nilpotent complex matrices with the same rank and same minimal polynomial. Prove they are similar.