Algebra Qualifying Exam Fall 1996

Directions: *Write each solution on a separate page. Please submit solutions in the same order as the questions. Write the last four digits of your student ID (social security number) at the top of each page. Do not write your name on the paper. All steps must be justified by computation or explanation. Greater weight may be given to one whole (correct) solution than to several incomplete solutions. To demonstrate adequate breadth, significant work must be done from each of Part I and Part II.*

Part I

1. Determine the invariant factors of the group of units of the ring of integers modulo 85.

2. Consider the alternating group A_5 on 5 letters.

   (a) Determine the sizes of the conjugacy classes of A_5.

   (b) Show that A_5 is simple.

3. Let F be a finite field. Let V be a one-dimensional vector space over F.

   (a) Show that each finite-dimensional F-vector space is isomorphic to a direct power of V.

   (b) Show that the coproduct (weak direct sum) of countably (infinitely) many copies of V is not isomorphic to a (possibly infinite) direct power of V.

4. Let F be a field of positive characteristic p.

   (a) Show that $\varphi : F \to F; x \mapsto x^p$ is a ring homomorphism.

   (b) If F is finite, show that φ is an automorphism of F.

   (c) Give an example of a field F for which φ is not surjective.

5. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an automorphism of the field of real numbers that leaves each rational number fixed. Show that φ also leaves each irrational number fixed.

Part II

6. Let A and G be real matrices of sizes $m \times n$ and $n \times m$ respectively. Let $B = \{b \in \mathbb{R}^m | \exists x \in \mathbb{R}^n \ . \ Ax = b\}$. For b in B, let $\mathcal{S}(b) = \{x \in \mathbb{R}^n | Ax = b\}$. Prove that the following conditions are equivalent:

   (1) $AGA = A$;

   (2) $\forall b \in B, \ Gb \in \mathcal{S}(b)$;

   (3) $\forall b \in B, \ \mathcal{S}(b) = \{Gb + (GA - I_n)z | z \in \mathbb{R}^n\}$.
   [In (3), I_n denotes the $n \times n$ identity matrix.]

7. For an $m \times n$ real matrix A, define Ker $A = \{x \in \mathbb{R}^n | Ax = 0\}$ and Im $A = \{Ax | x \in \mathbb{R}^n\}$. Show that:

   (1) Ker $A = [\text{ Im } (A^T)]^\perp$ and Im $A = [\text{Ker } (A^T)]^\perp$;

   (2) $\forall b \in \mathbb{R}^m, \ \exists x \in \mathbb{R}^n \ . \ A^T A x = A^T b$;

   (3) $\forall b \in \mathbb{R}^m, \ \exists x \in \mathbb{R}^n \ . \ \forall y \in \mathbb{R}^n, \ \|Ax - b\| \le \|Ay - b\|$.
   [Here, $\|z\| = \sqrt{z^T z}$ for z in $\mathbb{R}^m$.]

8. Let A and B be 2×2 complex matrices such that $tr\ A = tr\ B$, $tr\ A^2 = tr\ B^2$, and $tr\ AA^* = tr\ BB^*$. Prove that A and B are unitarily equivalent.

9. For each of the following cases, determine whether a matrix A exists with minimal polynomial $t^2 + 2t + 3$. In each case, either explain why such a matrix cannot exist, or else give an example of such a matrix.

   (a) A is a 3×3 real matrix.

   (b) A is a 2×2 real matrix.

   (c) A is a 3×3 complex matrix.

10. Let x and y be distinct vectors in $\mathbb{C}^n$. Show that the following conditions are equivalent:

    (1) There exists w in $\mathbb{C}^n$ such that the Householder transformation $T_w = I_n - \frac{2}{w*w} ww^*$ satisfies $T_w(x) = y$;

    (2) $\langle x, x \rangle = \langle y, y \rangle$ and $\langle x, y \rangle$ is real.
    [Here, the inner product in $\mathbb{C}^n$ is given by $\langle x, y \rangle = y^* x$.]