Algebra Qualifying Exam
August 21, 2008

Directions: Write the problem number and the last four digits of your student ID number at the top of each page. Do not write your name on your paper. Write each solution on a separate page. Submit solutions in the same order as the questions. All the steps must be justified by computation or explanation. Greater weight will be given to one whole (correct) solution than to two error-free but incomplete solutions. To demonstrate adequate breadth, significant work must be done from each of Part I and Part II.

Every effort is made to proofread the exam, but misprints may occur. If you believe that a problem has been stated incorrectly, check with the proctor and indicate your interpretation in the solution. Do not interpret the problem in a way that it becomes trivial.

Part I

1. Suppose $S$ and $T$ are subgroups of a group $G$. Let $S \lor T$ be the subgroup generated by $S \cup T$. Prove that if either $S$ or $T$ is normal, then $S \lor T = ST = TS$.

2. Show that the quaternion group $Q_8 = \langle r, s \mid r^4 = 1, s^2 = r^2, srs = r^{-1} \rangle$ is not isomorphic to a subgroup of the symmetric group $S_6$.

3. Let $R = \mathbb{Z}[\sqrt{-2}]$ and $N : R \to \mathbb{Z}_{\geq 0}$ be the map given by

$$N(a + b\sqrt{-2}) = a^2 + 2b^2,$$

for any $a + b\sqrt{-2} \in R$.

(a) Show that for any $\alpha, \beta \in R$, $N(\alpha \cdot \beta) = N(\alpha)N(\beta)$.

(b) Show that $R$ is a Euclidean Domain.

4. Let $G$ be a group of permutations. If $G$ contains an odd permutation, show that $G$ has a normal subgroup of index 2.

5. Show that the ring $\mathbb{R}$ of real numbers has no nontrivial ring automorphisms.

(Part II is on next page.)
Part II

6. Show that if a matrix \( A \in SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\} \) with \(|\text{tr}(A)| < 2\), then there is a nonzero integer \( n \) such that \( A^n = I \).

7. Let \( A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ -2 & 0 & -1 & 2 \\ -2 & -2 & 0 & 1 \end{pmatrix} \). Find the rational normal form and Jordan normal form of \( A \).

8. An \( n \times n \) real matrix \( A = [a_{ij}] \) is said to be row-stochastic if all its entries are non-negative, and
\[
\sum_{j=1}^{n} a_{ij} = 1
\]
for each \( 1 \leq i \leq n \). Show that each orthogonal row-stochastic matrix is a permutation matrix.

9. Let \( A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} \). For each positive integer \( n \), find \( a_n, b_n \) such that \( A^n = a_n A + b_n I_2 \).

10. Let \( A \in M_n(\mathbb{C}) \). Prove that the following conditions are equivalent:
   (a) \( A \) is normal;
   (b) \( A \) commutes with \( A^*A \);
   (c) \( \text{tr}[(A^*A)^2] = \text{tr}[(A^*)^2 A^2] \).