ALGEBRA QUALIFYING EXAMINATION

AUGUST 2006

Directions: Write the problem number and the last four digits of your student ID number at the top of each page. Do not write your name on your paper. Write each solution on a separate page. Submit solutions in the same order as the questions. All the steps must be justified by computation or explanation. Greater weight will be given to one whole (correct) solution than to two error-free but incomplete solutions. To demonstrate adequate breadth, significant work must be done from each of Part I and Part II.

Part I

(1) Let a and b be members of a finite abelian group. Prove that if $\gcd(|a|, |b|) = 1$, then $|ab| = |a||b|$. (Here, $|a|$ denotes the order of a in the group.)

(2) Prove that the polynomial ring $\mathbb{Z}[x]$ is not a PID. *Only displaying a non-principal ideal of* $\mathbb{Z}[x]$ *is not sufficient.*

(3) Which of the following groups of order 8 are isomorphic to subgroups of S_5? Explain your answer. $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, D (the dihedral group of order 8), Q (the quaternion group of order 8).

(4) Let G be a finite group and K a normal subgroup of G. If $|K|$ and $[G : K]$ are relatively prime, prove that K is the unique subgroup of G having order $|K|$.

(5) Prove or disprove: the rings $\mathbb{Q}[x]/(x^2 - 25)$ and $\mathbb{Q} \times \mathbb{Q}$ are isomorphic.

Part II

(6) Let

$$A = \begin{bmatrix} 7 & -8 & -3 & -5 \\ 1 & 1 & 0 & -1 \\ 2 & -5 & -1 & -2 \\ 4 & -7 & -3 & -2 \end{bmatrix}.$$

The characteristic polynomial of A is $(x - 2)^3(x + 1)$. Find (with justification) the rational canonical form (using elementary divisors) of A.

(7) Let $A \in \mathbb{C}^{n\times n}$. Define a product $\langle \cdot, \cdot \rangle_A : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ by $\langle v, w \rangle_A = w^* A v$. Prove: If the product $\langle \cdot, \cdot \rangle_A$ is an inner product, then A is Hermitian and has positive eigenvalues.

(8) Let A be an $n \times n$ normal complex matrix, and $H = AA^* = [h_{ij}]$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A. Show that

$$\max\{|h_{ii}| : i = 1, 2, ..., n\} \leq \max\{|\lambda_i|^2 : i = 1, 2, ..., n\}.$$

(9) Let A be an $n \times n$ real matrix with distinct real eigenvalues. Prove there exist $n \times n$ real matrices B, C such that $A = B^2 - C^2$ and $BC = CB = 0$.

(10) Let V be a finite dimensional vector space, W a subspace, and f a linear functional on W. Prove that there is a linear functional g on V which agrees with f on W.