ALGEBRA QUALIFYING EXAMINATION

AUGUST 20, 2005

Directions: Write the problem number and the last four digits of your student ID number at the top of each page. Do not write your name on your paper. Write each solution on a separate page. Submit solutions in the same order as the questions. All the steps must be justified by computation or explanation. Greater weight will be given to one whole (correct) solution than to two error-free but incomplete solutions. To demonstrate adequate breadth, significant work must be done from each of Part I and Part II.

Part I

1. Show that any group of order 45 has an element of order 15.
2. Suppose G is a group with exactly 3 elements of order 2. Prove that G is not simple. Hint: Consider the homomorphism $G \to S_3$ obtained by conjugation.
3. For each prime p that divides the order of A_6, give an example of a p-Sylow subgroup of A_6. Provide justification that your answers are correct.
4. Let $R = \mathbb{Q}[x]/(x^3 - x^2 - x + 1)$. Prove that R has exactly 6 ideals. Write down the six ideals of R and the lattice structure of the ideals of R.
5. Let $R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$. Let $N : R \to \mathbb{Z}_+$ given by
$$N(a + b\sqrt{-5}) = a^2 + 5b^2$$
where $\mathbb{Z}_+$ is the set of all non-negative integers.
   (a) Show that $z \in R$ is a unit if, and only if, $N(z) = 1$.
   (b) Show that the four elements $1 \pm \sqrt{-5}, 2, 3$ are irreducible over R.
   (c) Show that R is not a PID.

Part II

6. Show that if $A \in M_n(\mathbb{C})$ is similar to a unitary matrix, then A^{-1} is similar to A^*.
7. Let A, B be 3×3 complex matrices. Prove that A and B are similar if, and only if, they have the same characteristic polynomials and minimal polynomials.
8. Let $H = [h_{ij}] \in M_n(\mathbb{C})$ be a Hermitian matrix with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. For each statement below, prove the statement is true or give an example showing the statement need not be true.
   (a) All diagonal entries h_{ii} are real.
   (b) For each $i = 1, 2, ..., n$, $\lambda_1 \le h_{ii} \le \lambda_n$.
   (c) For all $i, j \in \{1, 2, ..., n\}$, $i \ne j$, $\lambda_1 \le \det \begin{bmatrix} h_{ii} & h_{ij} \\ h_{ji} & h_{jj} \end{bmatrix} \le \lambda_n$.
9. Suppose $A \in M_n(F)$ and $x \in F^n$ satisfy $A^k x \ne 0$ and $A^{k+1}x = 0$. Prove that $\{x, Ax, \ldots, A^k x\}$ is linearly independent.
10. Let $A \in M_n(\mathbb{Q})$ satisfy $A^5 = I$. Prove: If 1 is not an eigenvalue of A then n must be a multiple of 4. You may assume $x^4 + x^3 + x^2 + x + 1$ is irreducible in $\mathbb{Q}[x]$.