ALGEBRA QUALIFYING EXAM, FALL 2004

Directions:

Write the problem number and the last four digits of your student ID number at the top of each page. Do not write your name on your paper. Write each solution on a separate page. Submit solutions in the same order as the questions.

All steps must be justified by computation or explanation. Greater weight will be given to one whole (correct) solution than to two error-free but incomplete solutions. To demonstrate adequate breadth, significant work must be done from each of Part I and Part II.

Part I

1. Let θ be an automorphism of the symmetric group S_3. Prove that there is an element g of S_3 such that $\theta(x) = g^{-1}xg$ for all x in S_3.

2. Let p be a prime divisor of the order of a finite group G. Prove that G contains an element of order p.

3. Let D be a principal ideal domain. Prove that each irreducible element of D is prime.

4. Let F be a subfield of a field K. Suppose that $p(X)$ and $q(X)$ are relatively prime in $F[X]$. Prove that $p(X)$ and $q(X)$ are relatively prime in $K[X]$.

5. Let $\mathbb{D}$ be the subset $\{m \cdot 2^n \mid m, n \in \mathbb{Z}\}$ of the set $\mathbb{Q}$ of rationals.

   (a) Prove that $\mathbb{D}$ is a subgroup of the additive group $\mathbb{Q}$.

   (b) Prove that the groups $\mathbb{D}$ and $\mathbb{Q}$ are not isomorphic.

[Continuation on other side.]

Part II

6. Show that each complex matrix A is similar to its transpose.

7. Let A be an $n \times n$ matrix over a field K. Prove that there exists an $n \times n$ matrix B such that $ABA = A$ and $BAB = B$.

8. Let A be an $n \times n$ matrix over a field K. Suppose that A has n distinct eigenvalues in K, and that B is an $n \times n$ matrix over K that commutes with A. Prove that there is a polynomial $p(X)$ in $K[X]$ such that $p(A) = B$.

9. Let S be an $n \times n$ skew-symmetric real matrix. Prove that $(I_n + S)\mathbf{x} = \mathbf{0}$ has no non-zero real solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} .$$

10. Show that two Hermitian matrices are similar if and only if they are unitarily similar.