ALGEBRA QUALIFYING EXAM FALL 2000

DIRECTIONS: Write each solution on a separate page. Please submit solutions in the same order as the questions. Write the last four digits of your student ID (social security number) at the top of each page. Do not write your name on the paper.

All steps must be justified by computation or explanation. Greater weight may be given to one whole (correct) solution than to several incomplete solutions. To demonstrate breadth, significant work must be done from each of Part I and Part II.

PART I

1. Let S be the set of rational numbers in the open unit interval $]0,1[$. For elements p_i/q_i of S in their lowest terms, define

$$\frac{p_1}{q_1} \oplus \frac{p_2}{q_2} = \frac{p_1 + p_2}{q_1 + q_2}.$$

   a) Show that $\oplus$ is a well-defined, commutative operation on S.

   b) Criticize the following "proof" that $\oplus$ is associative:

$$\left(\frac{p_1}{q_1} \oplus \frac{p_2}{p_2}\right) \oplus \frac{p_3}{q_3} = \frac{p_1 + p_2 + p_3}{q_1 + q_2 + q_3} = \frac{p_1}{q_1} \oplus \left(\frac{p_2}{q_2} \oplus \frac{p_3}{q_3}\right).$$

2. What is the largest size of a set S of abelian groups of order 10,000 such that no two elements of S are isomorphic? Justify your answer.

3. For each natural number n, let A_n be the group of integers under addition. Give a careful proof that the quotient $\prod_{n\in\mathbb{N}} A_n / \bigoplus_{n\in\mathbb{N}} A_n$ of the direct product of the A_n by the direct sum of the A_n contains a subgroup F that is free.

4. Prove that no group of order 100 is simple.

5. Let F be the free group over the set $\mathbb{N}$ of natural numbers. Give a detailed proof that its automorphism group $\mathrm{Aut}(F)$ is uncountable.

PART II

6. Let V be a vector space over an infinite field. Suppose $v_1, \ldots, v_m$ are nonzero vectors in V. Prove there exists a linear functional f on V such that $f(v_i) \neq 0$ for all $i = 1, \ldots, m$.

7. Let A be an $n \times n$ complex matrix. Define $T_A : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $T_A(B) = AB - BA$. If $\lambda \neq 0$ is an eigenvalue of T_A with eigenvector C, prove that C is nilpotent.

8. a) A and B are 3×3 complex matrices such that $A^2 = B^3 = 0$ and rank A = rank B. Prove A and B are similar.

   b) Give an example to show that a) is false if A and B are 4×4 matrices.

9. Let $A \in M_n(\mathbb{C})$. Prove that if λ is an eigenvalue of A^2 then there is an eigenvalue μ of A such that $\mu^2 = \lambda$.

10. Let A be an $n \times n$ normal complex matrix. Prove that A^* is polynomial in A.