The minimum number of distinct eigenvalues among the symmetric matrices with a given graph

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Abstract: A classic result is that the number of distinct eigenvalues of a graph of diameter d is at least $d+1$. More generally, let G be a graph with n vertices, and let $S(G)$ denote the set of all symmetric n by n matrices, $A = [a_{ij}]$, with the property that for off-diagonal entries, a_{ij} is nonzero if and only if G has an edge from i to j. The traditional proof of the classic result extends to $A \in S(G)$. Thus, if $A \in S(G)$, then A has at least $d+1$ distinct eigenvalues. Saiago and Johnson recently conjectured that if T is a tree of diameter d, then there exists a matrix in $S(T)$ with exactly $d+1$ distinct eigenvalues. More recently, Barioli and Fallat have disproved the conjecture with a tree of diameter 7, such that every matrix in $S(T)$ has at least 9 distinct eigenvalues. We use the Smith Normal Form for matrices with polynomial entries, to give an inifinite family of trees T of diameter d such that every $A \in S(T)$ has at least $(9/8)d$ distinct eigenvalues. We also show that for each d, there is an integer $f(d)$, such that every tree T of diameter d (but an arbitrary number of vertices) there exists a matrix $A \in S(T)$ with at most $f(d)$ distinct eigenvalues. Finally, we discuss bounds on $f(d)$ for small d.