OPTIMAL CONTROL OF STOCHASTIC BURGERS' EQUATION WITH RANDOM FORCING USING WEINER CHAOS EXPANSION

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Introduction . Control of nonlinear stochastic partial differential equations (SPDEs) has acquired significant attention in recent years. But due to the nonlinearity and randomness of the system, such problem is very challenging and difficult to handle. A typical example in this field is the control of stochastic Burgers' equations (SBEs) with random forcing, which is presented in [2] as a step to develop a method for control of turbulent flows.

On the other hand, Wiener chaos expansion (WCE) [11] provides a new direction to solve the nonlinear SPDEs numerically, see e.g. [7], [9], [1], [10], [12]. Especially for some nonlinear SPDEs, e.g. SBEs and Navier-Stokes equations, with random forcing, WCE may be a more efficient and accurate numerical method than Monte Carlo simulation[8]. Actually the WCE is the Fourier expansion in the probability space,

$$u(x,t,\omega) = \sum_{\alpha \in \mathcal{J}} u_\alpha(x,t) H_\alpha(\omega),$$

where $\mathcal{J}$ denotes the set of multi-indices $(\alpha_1, \alpha_2, \cdots)$ (where $\alpha_i \in \mathbb{N}_0$) with only finitely many components are nonzero, each u_{α_i} is deterministic and called WCE coefficient, and $\{H_\alpha\}$ is an orthonormal basis of the probability space and hence is stochastic variable. In other words, by expanding u to the WCE form, we can decompose the stochastic function to the deterministic part and randomness. Therefore, for a solution $u(x,t,\omega)$ of a nonlinear SPDE, after plugging the expansion of $u(x,t,\omega)$ into the equation, we can establish a deterministic partial differential equation (PDE) system for the WCE coefficients u_α, which can be solved efficiently by some deterministic numerical methods. Furthermore, due to the rapid decay of the magnitude of the coefficients, we can reduce our deterministic system to a smaller one.

Now we are concerned with the application of WCE method in optimal control problems governed by the SBEs with a random forcing, and take Wick type SBEs [5] as our testing model. Particularly, we shall study the distributed and boundary control problems involving the SBEs

$$u_t + u \diamond u_x = \nu u_{xx} + \dot{W},$$

where $\diamond$ is the Wick product, which can be considered as a regularization of the ordinary multiplication [4], W is a Brownian motion and $\nu > 0$ denotes a viscosity parameter. The uniqueness and existence of the solutions for such type SBEs can be proved by using a Wick version Cole-Hopf transformation. By implementing the WCE method and an adjoint-based iterative algorithm [3], we derive a variational formulation for the control problems. Figure. 0.1 gives the outline of our optimization algorithm.

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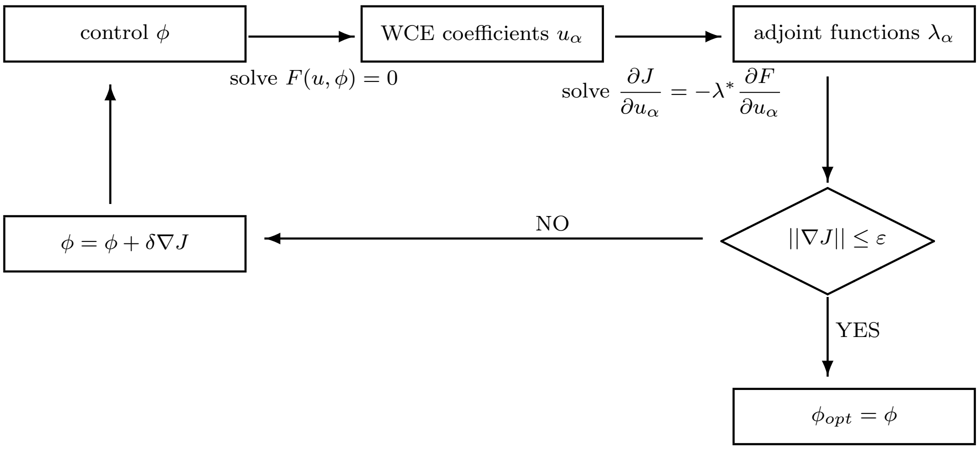
FIG. 0.1. *The outline of the solving algorithm*



where $F(u,\phi)=0$ denotes the SBEs with random forcing, ϕ is the control, J is the objective functional, λ is the Lagrange multiplier function and λ_α is the multiplier function corresponding to H_α.

One problem raised naturally is the number of the WCE coefficients in our truncation, which is determined by the accuracy we want to acquire, and for the same accuracy, a long time computation would also need more coefficients. Some error estimates of WCE solutions for the nonlinear PDEs may be found at [6], and in our numerical test, we use fifth order WCE with 8 Gaussian random variables as our approximation. Actually for a Nth order with K Gaussian random variables approximation, the number of WCE coefficients would be $\binom{K+N}{K}$, thus the number would increase dramatically as N and K increase, which will leads to an unacceptable computational cost, even for a fifth order approximation. To avoid this, we firstly use the technique of sparse truncation, see e.g. [8] and [10], which can reduce the number of coefficients efficiently without losing much accuracy. On the other hand, we employ our algorithm to a lower order WCE approximation to predict our controls, next use the results as the initial control guess for the higher-order WCE approximation. By comparing the results achieved by third order WCE approximation with only 7 coefficients to fifth order truncation with 117 coefficients, we found that the difference between the controls is surprisingly small, which actually shows the stability of our optimal solution from another point of view, and also implies that even for the order of WCE truncation higher than five, the optimal solution would be quite close to the results we achieve. This aspect can be explained roughly that our optimal solutions are acquired by controlling the most important coefficients, and due to the rapid decay of the coefficients, the rest coefficients have not much impact.

Furthermore, the algorithm can also be implemented to the control problems subject to general stochastic Burgers equations with an random source, like the problem in [2], one only need to change the product of WCE from Wick type to general one.

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