Part 4 Numerical ranges and quantum computing

The numerical range and the numerical radius of $A \in M_n$ are defined as

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\} \quad \text{and} \quad r(A) = \max\{|\mu| : \mu \in W(A)\}.$$ 

These concepts are useful in the study of matrices. There are many generalizations motivated by applications. We discussed some basic properties and selected generalizations useful in quantum computing.

1 The classical numerical range

Proposition 1.1 Let $A \in M_n$.

1. $W(U^*AU) = W(A)$ for any unitary $U \in M_n$.
2. $W(A + cI) = W(A) + c$ for any $c \in \mathbb{C}$.
3. $W(cA) = cW(A)$ for any $c \in \mathbb{C}$.
4. $\sigma(A) \subseteq W(A)$.
5. $W(A + B) \subseteq W(A) + W(B)$ for any $B \in M_n$.
6. $W(A \oplus B) = \operatorname{conv} [W(A) \cup W(B)]$ for any $B \in M_m$.
7. $W(A) = \operatorname{conv} \{a_1, \ldots, a_n\}$ if $A$ is normal with eigenvalues $a_1, \ldots, a_n$.

Theorem 1.2 For any $A \in M_n$, $W(A)$ is a compact convex set in $\mathbb{C}$. If $A \in M_2$ then $W(A)$ is an elliptical disk with the eigenvalues $a_1, a_2$ of $A$ as foci and $\gamma = \sqrt{\operatorname{tr}(A^*A) - |a_1|^2 - |a_2|^2}$ as minor axis.

Theorem 1.3 Let $A \in M_n$.

1. $W(A) = \{\mu\}$ if and only if $A = \mu I$.
2. $W(A) \subseteq a\mathbb{R} + b$ if and only if $A = aH + bI$ with $H = H^*$.
3. $A$ is unitary if and only if $A$ is invertible such that both $W(A)$ and $W(A^{-1})$ are subsets of the closed unit disks.

Theorem 1.4 Let $A \in M_n$. Then $\operatorname{Re}(W(A)) = W((A + A^*)/2)$. Consequently,

$$W(A) = \{\mu \in \mathbb{C} : e^{it}\mu + e^{-it}\bar{\mu} \leq \lambda_1(e^{it}A + e^{-it}A^*), t \in [0, 2\pi]\}.$$
Theorem 1.5 Let $A \in M_3$ be a unitarily reducible matrix or $A \in M_2$. Then $B \in M_n$ satisfies $W(B) \subseteq W(A)$ if and only if $B = X^*(A \otimes I_m)X$ for some matrix $X$ of appropriate size such that $X^*X = I_n$.

Theorem 1.6 Let $A$ and $B$ be square matrices. Define the function $\Phi$ from span $\{I, A, A^*\}$ to span $\{I, B, B^*\}$ by $\Phi(aI + bA + cA^*) = aI + bB + cB^*$.

(a) Then $W(B) \subseteq W(A)$ if and only if $\Phi$ is a positive linear map.
(b) The matrix $B$ is a compression of $A \otimes I$ if and only if $\Phi$ is a completely positive linear map.

Theorem 1.7 Let $A \in M_n$. Then

$$\rho(A) \leq r(A) \leq \|A\| \leq 2r(A)$$

and

$$r(A^k) \leq r(A)^k, \quad k = 1, 2, \ldots.$$ 

(a) The equality $\rho(A) = r(A)$ holds if and only if $A$ is unitarily similar to a matrix of the form $[\mu] \oplus A_2$ such that $|\mu| = r(A)$.

(b) The equality $\rho(A) = \|A\|$ holds if and only if $r(A) = \|A\|$. This happens if and only if $A$ is unitarily similar to a matrix of the form $[\mu] \oplus A_2$ such that $|\mu| = \|A\|$.

(c) The equality $\|A\| = 2r(A)$ holds if and only if $A/r(A)$ is unitarily similar to a matrix of the form $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \oplus A_2$ with $r(A_2) \leq 1$.

Theorem 1.8 Let $A, B \in M_n$. Then

$$r(AB) \leq 4r(A)r(B).$$

If $AB = BA$, then

$$r(AB) \leq 2r(A)r(B).$$

Problem 1.9 Determine the best (smallest) constant $\gamma$ such that $r(AB) \leq \gamma r(A)\|B\|$ for $A$ and $B$ such that $AB = BA$.

Problem 1.10 Determine the best (smallest) constant $\gamma$ such that

$$\|p(A)\| \leq \gamma \max\{|p(\mu)| : \mu \in W(A)\}$$

for any complex polynomial $p(z)$. 

2
2 The higher rank numerical range

In connection to quantum error correction, see the appendix, researchers consider the rank $k$-numerical range of $A \in M_n$ defined by

$$\Lambda_k(A) = \{ \mu \in \mathbb{C} : \text{there is } P \in \mathcal{P}_k \text{ such that } PAP = \mu P \},$$

where $\mathcal{P}_k$ is the set of rank $k$-orthogonal projections in $M_n$.

**Theorem 2.1** Let $A \in M_n$ and $1 \leq k \leq n$.

1. For any $a, b \in \mathbb{C}$, $\Lambda_k(aA + bI) = a\Lambda_k(A) + b$.
2. For any unitary $U \in M_n$, $\Lambda_k(U^*AU) = \Lambda_k(A)$.
3. If $B \in M_r$ is a compression of $A$ with $r \geq k$, then $\Lambda_k(B) \subseteq \Lambda_k(A)$.
4. Suppose $n < 2k$. The set $\Lambda_k(A)$ has at most one element.

**Theorem 2.2** Let $w = e^{i2\pi/3}$ and

$$B = I_{k-1} \oplus wI_{k-1} \oplus w^2I_{k-1}.$$  

If $n \leq 3k - 3$, then for any $(3k - 3) \times n$ matrix $X$ satisfying $X^*X = I_n$, $\Lambda_k(X^*BX) = \emptyset$.

If $n \geq 3k - 2$ then $\Lambda_k(A)$ is non-empty for any $A \in M_n$.

**Theorem 2.3** Let $A \in M_n$. Then $\Lambda_k(A) = \Omega_k(A)$, where

$$\Omega_k(A) = \bigcap_{\xi \in [0,2\pi)} \{ \mu \in \mathbb{C} : e^{i\xi}\mu + e^{-i\xi}\overline{\mu} \leq \lambda_k(e^{i\xi}A + e^{-i\xi}A^*) \}.$$  

In particular, if $A \in M_n$ is a normal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$\Lambda_k(A) = \bigcap_{1 \leq j_1 < \cdots < j_n-k+1 \leq n} \text{conv} \{ \lambda_{j_1}, \ldots, \lambda_{j_n-k+1} \}.$$  

**Corollary 2.4** For any $A \in M_n$ and $1 \leq k \leq n$, $\Lambda_k(A)$ is convex.
3 The joint higher rank numerical range

**Definition 3.1** Let \( A_1, \ldots, A_m \in M_n \). The joint rank-\( k \) numerical range of \( A = (A_1, \ldots, A_m) \) is defined by

\[
\Lambda_k(A) = \{ (a_1, \ldots, a_m) : \text{there is } P \in \mathcal{P}_k \text{ such that } PA_jP = a_jP, \ j = 1, \ldots, m \},
\]

where \( \mathcal{P}_k \) is the set of rank \( k \) orthogonal projections in \( M_n \).

**Remark 3.2** If \( A_j = H_j + iG_j \) with \( H_j = H_j^* \) and \( G_j = G_j^* \), then \( \Lambda_k(A_1, \ldots, A_m) \subseteq \mathbb{C}^{1 \times m} \) can be identified as \( \Lambda_k(H_1, G_1, \ldots, H_m, G_m) \subseteq \mathbb{R}^{1 \times 2m} \). So, we may focus on the joint rank \( k \)-numerical range of Hermitian matrices.

**Proposition 3.3** Suppose \( A_1, \ldots, A_m \in H_n \). Let \( T = (t_{ij}) \in M_m(\mathbb{R}) \) and \( (c_1, \ldots, c_m) \) be a real vectors. If \( B_j = c_jI + \sum_{i=1}^m t_{ij}A_i \), then

\[
\Lambda_k(B_1, \ldots, B_m) = \{ (c_1, \ldots, c_m) + (a_1, \ldots, a_m)T : (a_1, \ldots, a_m) \in \Lambda_k(A_1, \ldots, A_m) \}.
\]

**Theorem 3.4** Let \( A_1, \ldots, A_m \in H_n \). Then \( W(A_1, \ldots, A_m) \) is convex if

(a) \( \text{span} \{ I, A_1, \ldots, A_m \} \) has dimension at most 3, or

(b) \( n \geq 3 \) and \( \text{span} \{ I, A_1, \ldots, A_m \} \) has dimension at most 4.

**Example 3.5** Let

\[
B_1 = I_2 \oplus 0_{n-2}, \ B_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus 0_{n-2}, \ B_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 0_{n-2}, \ B_4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \oplus 0_{n-2}.
\]

Then \( W(B_1, B_2, B_3, B_4) \) is not convex.

**Theorem 3.6** Let \( A_1, A_2, A_3 \in H_n \) be such that \( \text{span} \{ I, A_1, A_2, A_3 \} \) has dimension 4. Then there is \( A_4 \) such that \( W(A_1, A_2, A_3, A_4) \) is not convex.

There are many problems on \( \Lambda_k(A_1, \ldots, A_m) \) under active research.

**Problem 3.7** Let \( A_1, \ldots, A_m \in H_n \). For \( k > 1 \) the set \( \Lambda_k(A_1, \ldots, A_m) \) may be empty. Determine the minimum \( n \) (in terms of \( m \) and \( k \)) so that \( \Lambda_k(A_1, \ldots, A_m) \) is always non-empty for \( A_1, \ldots, A_m \in H_n \).

**Theorem 3.8** For \( m, k \geq 1 \), let

\[
n(m, k) = \begin{cases} 2 \cdot 3^{\frac{m-1}{2}}(k-1) + 1 & \text{when } m \text{ is odd} , \\ 3^{\frac{m}{2}}(k-1) + 1 & \text{when } m \text{ is even} . \end{cases}
\]

Then \( \Lambda_k(A_1, \ldots, A_m) \) is non-empty for all \( A_1, \ldots, A_m \in H_n \).
Example 3.9 For $m, k > 1$, let $n = (m+1)(k-1)$. Suppose $A_1 = I_{k-1} \oplus 0_{k-1} \oplus -I_{(m-1)(k-1)}$ and

$$A_j = I_j(k-1) \oplus 0_{(m+1-j)(k-1)}, \quad j = 2, \ldots, m.$$ 

Then $\Lambda_k(A_1, \ldots, A_m) = \emptyset$.

Proposition 3.10 Suppose $A_1, \ldots, A_m \in H_n$ are diagonal matrices. If $n > m+1$, then $\Lambda_2(A_1, \ldots, A_m) \neq \emptyset$.

Problem 3.11 Can we extend the above result to general Hermitian matrices $A_1, \ldots, A_m$?

Theorem 3.12 Let $A = (A_1, \ldots, A_m) \in H_n^m$. If $(a_1, \ldots, a_m) \in \Lambda_{\hat{k}}(A)$, where $\hat{k} \geq (m+2)k$ if $k > 1$ and $\hat{k} \geq (m+1)/2$ if $k = 1$. Then $\Lambda_k(A_1, \ldots, A_m)$ is star-shaped with $(a_1, \ldots, a_m)$ as a star-center. Consequently, $\text{conv} \Lambda_{\hat{k}}(A_1, \ldots, A_m)$ is a compact convex subset of $\Lambda_k(A)$.

Problem 3.13
1. Determine the minimum $n$ such that $\Lambda_k(A_1, \ldots, A_m)$ is star-shaped for any $A_1, \ldots, A_m \in H_n$.
2. Determine the condition on $A_1, \ldots, A_m \in H_n$ so that $\Lambda_k(A_1, \ldots, A_m)$ is convex.
3. Determine a “large” convex subset of $\Lambda_k(A_1, \ldots, A_m)$.

4 The $C$-numerical range and quantum control

Definition 4.1 Let $C \in M_n$. The $C$-numerical range and the $C$-numerical radius of $A \in M_n$ are defined by

$$W_C(A) = \{ \text{tr}(CU^*AU) : U \text{ is unitary} \}$$

and

$$r_C(A) = \max\{ |\mu| : \mu \in W_C(A) \}.$$ 

Note that the $C$-numerical radii are the building blocks for USI norms on $M_n$.

Theorem 4.2 Suppose $C = aI + bR$ where $R$ is Hermitian or rank one. Then $W_C(A)$ is convex for any $A \in M_n$.

Definition 4.3 A matrix $C$ is a block shift operator if it is unitarily similar to a block matrix $(C_{ij})_{1 \leq i,j \leq m}$ such that $C_{11}, \ldots, C_{mm}$ are square matrices, and $C_{ij} = 0$ whenever $i \neq j + 1$.

Theorem 4.4 Suppose $C = aI + R$ where $R$ is a block shift operator. Then $W_C(A)$ is a circular disk for any $A \in M_n$. 

5
Problem 4.5 Characterize matrices $C \in M_n$ such that $W_C(A)$ is convex for all $A \in M_n$.

Definition 4.6 Let $C \in M_n$ have eigenvalues $c_1, \ldots, c_n$. Define the $C$-spectral radius and $C$-spectral norm of $A \in M_n$ by

$$
\rho_C(A) = \max \left\{ \left| \sum_{j=1}^n c_{i_j} \lambda_j(A) \right| : (i_1, \ldots, i_n) \text{ is a permutation of } (1, \ldots, n) \right\},
$$

and

$$
\|A\|_C = \max \{ \text{tr}(CUAV) : U, V \text{ are unitary} \}.
$$

Theorem 4.7 Let $C \in M_n$ have singular values $c_1 \geq \cdots \geq c_n$. Then

$$
\|A\|_C = \sum_{j=1}^n c_js_j(A).
$$

Note that the $C$-spectral norms are the building blocks of UI norms on $M_n$.

In quantum control, it is important to determine

$$
\min \{ \|C - U^*BU\| : U \text{ is unitary} \}
$$

for two given (nilpotent) matrices $C$ and $A$ arising from some quantum mechanical systems.

Note that

$$
\|C - U^*BU\|^2 = \|C\|^2 + \|A\|^2 - 2\text{Re}(\text{tr}(CU^*B^*U)).
$$

So, the problem reduces to finding

$$
\rho_C(B^*) = \max \{ \text{Re}(\text{tr}(CU^*B^*U)) : U \text{ is unitary} \}.
$$

Problem 4.8 Determine $r_{C_k}(A_k)$ for

$$
C_k = \begin{pmatrix} 0_{2^k} & 0_{2^k} \\ I_{2^k} & 0_{2^k} \end{pmatrix} \quad \text{and} \quad A_k = N_k \oplus N_k,
$$

where

$$
N_0 = (0), \quad N_k = \begin{pmatrix} N_{k-1} & 0 \\ I_{2^{k-1}} & N_{k-1} \end{pmatrix}.
$$

Here are some conjectured values:

<table>
<thead>
<tr>
<th>$k$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_{C_k}(A_k)$</td>
<td>$4(1 + \sqrt{3})$</td>
<td>$8(1 + \sqrt{3})$</td>
<td>$16(1 + \sqrt{3}) + 4\sqrt{5}$</td>
<td>$32(1 + \sqrt{3}) + 8\sqrt{5}$</td>
</tr>
</tbody>
</table>
Recently, researchers study the local $C$-numerical range and $C$-numerical radius with respect to a certain subgroup $S$ of the unitary group defined by

$$W_{S(C)}(A) = \{ \text{tr} (C U^* A U) : U \in S \}$$

and

$$r_{S(C)}(A) = \{ |\mu| : \mu \in W_{S(C)}(A) \}.$$ 

5 **Exercises**

1. Suppose $\mu \in \sigma(A)$ is a boundary point of $W(A)$. Show that $A$ is unitarily similar to $[\mu] \oplus A_2$.

2. Show that if $\mu \in W(A)$ satisfies $|\mu| = \|A\|$, then $A$ is unitarily similar to $[\mu] \oplus A_2$.

3. Show that if $A \in M_n$ and $W(A)$ is a convex polygon (with interior) with $n-1$ vertices, then $A$ is normal. For each $n \geq 5$, show that there is a non-normal matrix $B$ such that $W(B)$ is a convex polygon with $n-2$ vertices.

4. Let

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. $$

Show that

$$W(A_1, A_2, A_3) = \{ (a, b, c) : a, b, c \in \mathbb{R}, a^2 + b^2 + c^2 = 1 \}. $$

5. Give a complete description of $\Lambda_2(A)$ for a normal matrix $A \in M_4$ in terms of its eigenvalues.

6. Show that if $A \in M_n$ has rank less than $k$, then $\Lambda_k(A) = \{ 0 \}$.

7. Suppose $n \geq 2k$. There is $A \in M_n$ such that $\Lambda_k(A)$ is the unit circular disk.

8. Suppose $n \geq 2k + m$. There is $A \in M_n$ such that $\Lambda_k(A)$ is a regular $m$-side polygons.

9. If $W_C(A)$ is always a circular disk centered at the origin, show that $C$ is a block shift operator.

If $W_C(A)$ is always a circular disk, can we conclude that $C = aI + R$ for a block shift operator $R$?
Appendix: Background of quantum error correction

In classical computing, information is stored as binary sequences. A length \( k \) sequence is encoded as a length \( n \) sequence, and then transmitted through a noisy channel. The received sequence can be correctly decoded provided there are fewer than \( r(n, k) \) error.

In quantum computing, information is stored in **quantum bits (qubits)**. Mathematically, a qubit is represented by a \( 2 \times 2 \) rank one projection
\[
Q = \frac{1}{2} \begin{pmatrix} 1 + z & x + iy \\ x - iy & 1 - z \end{pmatrix}
\]
with \( x^2 + y^2 + z^2 = 1 \). A state of \( K \)-qubits \( Q_1, \ldots, Q_K \) is represented by their tensor products in \( M_n \) with \( k = 2^K \). Again, a state of \( K \)-qubits is encoded as a state of \( N \)-qubits, and transmitted through a **quantum channel**, where a quantum channel for states of \( N \)-qubits is a **trace preserving completely positive linear map** \( \Phi : M_n \to M_n \) with \( n = 2^N \). By the result of Choi, there are \( T_1, \ldots, T_m \in M_n \) with \( \sum_{j=1}^m T_j^*T_j = I_n \) such that
\[
\Phi(X) = \sum_{j=1}^m T_jXT_j^*.
\] (1)

In this setting an quantum error correction code is a subspace \( V \) of \( \mathbb{C}^n \) such that the compression of \( \Phi \) on \( V \) is the identity map. By the result of Knill-Laflamme, this happens if and only if there are scalars \( \gamma_{ij} \) with \( 1 \leq i, j \leq r \) such that
\[
PT_i^*T_jP = \gamma_{ij}P, \quad 1 \leq i, j \leq m,
\]
where \( P \in M_n \) is an orthogonal projection of \( \mathbb{C}^n \) onto \( V \).

In connection to this, researchers study the joint rank-\( k \) numerical range of \( (A_1, \ldots, A_m) \) to be the set \( \Lambda_k(A_1, \ldots, A_m) \) of complex vectors \( (a_1, \ldots, a_m) \in \mathbb{C}^{m \times 1} \) for the existence of an rank-\( k \) orthogonal projection \( P \in M_n \) such that \( PA_jP = a_jP \) for \( j = 1, \ldots, m \).