
Before: \( X = A X \)

Now: \( \dot{X}(t) = A(t)X(t) \), where \( A(t) \) is for every \( t \in \mathbb{R} \) is a \( d \times d \) matrix.

If \( t \mapsto A(t) : \mathbb{R} \to \text{gl}(d, \mathbb{R}) \) is Lebesque integrable on every compact interval, then we consider

\[
x(t) = x_0 + \int_{t_0}^{t} A(s)x(s) \, ds, \quad s \in \mathbb{R}.
\]

Aim: understand the relations between \( A(t) \) and the behavior of solutions.

Conjecture:

\[
x(t) = e^{\int_{t_0}^{t} A(s) \, ds} x_0
\]

if \( d = 1 \) this is true

if \( d > 0 \) this is not true, because \( A(t_1) \) and \( A(t_2) \) may not commute

Ex. Let

\[
A(t) = \begin{cases} A_1 & \text{for } t \in [0, 1] \\ A_2 & \text{for } t > 1 \end{cases}
\]

with \( A_1 A_2 \neq A_2 A_1 \), then the formula does not work.

Conjecture:

If for all \( t \in \mathbb{R} \) the eigenvalues of \( A(t) \) have negative real parts, then all solutions will tend to zero as \( t \to \infty \)

Ex. \( x = A(t)x \) in \( \mathbb{R}^2 \) with

\[
A(t) = \begin{pmatrix}
1 - 4(\cos 2t)^2 & 2 + 2 \sin 4t \\
-2 + 2 \sin(4t) & 1 - 4(\sin 2t)^2
\end{pmatrix}
\]

\( A(t) \) is \( \pi \)-periodic

A solution is \( x(t) = \begin{pmatrix} e^t \sin 2t \\ e^t \cos 2t \end{pmatrix} \) (check!)

it is unbounded for \( t \to \infty \)

The eigenvalues:

\[
\det[A(t) - \lambda I] = \lambda^2 + 2 \lambda + 1 = (\lambda + 1)^2 \Rightarrow \text{eigenvalues are } \lambda_1 = \lambda_2 = -1.
\]
This phenomenon was known to Lyapunov in 1892.
Denote solutions by \( \varphi(t, t_0, x_0), t \in \mathbb{R} \).

Lyapunov exponent:

\[
\lim_{t \to \pm \infty} \frac{1}{t} \ln \| \varphi(t, t_0, x_0) \|
\]

in the example Lyapunov exponent = 1.

w.l.o.g. \( t_0 = 0 \) suffices.

A fundamental solution is a \( d \times d \)-matrix function \( X(t, t_0) \),
satisfying \( X(t) = A(t)X(t), X(t_0) = I \).
Then \( X(t) \) is invertible for every \( t \) and

\[
\varphi(t, t_0, x_0) = X(t, t_0)x_0 \quad \text{(generalization of } x = Ax, x(t_0) = x_0) \Rightarrow X(t, t_0) = e^{A(t-t_0)}, x(t) = X(t, t_0)x_0
\]

Formally:

\[
\varphi(t_0+s+t, t_0, x_0) = \varphi(t_0+s+t, t_0+s, \varphi(t_0+s, t_0, x_0)) - \text{cocycle property}
\]

Without loss of generality \( t_0 = 0 \) suffices:

\[
\lim_{t \to \pm \infty} \frac{1}{t} \ln \| \varphi(t, 0, \varphi(t_0, x_0)) \| = \lim_{t \to \pm \infty} \frac{1}{t} \ln \| \varphi(t, t_0, x_0) \|
\]

Claim: There are at most \( d \) different Lyapunov exponents.

Proof (Exercise)

Consequence: \( W_\lambda \) is a subspace of \( \mathbb{R}^d \) : \( x + y \in W_\lambda \) for \( x, y \in W_\lambda \)

\[
\lambda(\alpha x) = \lim_{t \to \pm \infty} \frac{1}{t} \ln \| \varphi(t, \alpha x) \| = \lim_{t \to \pm \infty} \frac{1}{t} \left[ \ln |\alpha| + \ln \| \varphi(t, x) \| \right] =
\]

\[
= \lim_{t \to \pm \infty} \frac{1}{t} \ln \| \varphi(t, x) \| = \lambda(x).
\]
$W_1, W_2, ..., W_i \leq \mathbb{R}^d \Rightarrow p \leq d$.

Let $v_1, ..., v_d$ be a basis of $\mathbb{R}^d$. Then the maximal Lyapunov exponent is attained on one of the $v_i$.

Let $x_0 \in \mathbb{R}^d$ with $\lambda(x_0)$ maximal.

\[ x_0 = \alpha_1 v_1 + ... + \alpha_d v_d \]

\[ \lambda(x_0) \leq \max \{ \lambda(\alpha_1 v_1), ..., \lambda(\alpha_d v_d) \} \Rightarrow \lambda(x_0) = \lambda(v_i) \text{ for some } i. \]

There is no decomposition into Lyapunov spaces for general time varying linear systems.

Some additional assumptions on $A(t)$ are needed.

### 7 Linear Skew-Product Flows

**Def.** A linear skew product flow is a map $\Phi: \mathbb{R} \times B \times \mathbb{R}^d \rightarrow B \times \mathbb{R}^d$ with $\Phi = (\Theta, \varphi)$ defined as follows:

\[ \Theta: \mathbb{R} \times B \rightarrow B \text{ satisfies } \Theta_{t+s} = \Theta_t \circ \Theta_s, \quad t, s \in \mathbb{R} \]

$\Theta_0 = \text{id}_B$ and $\varphi: \mathbb{R} \times B \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is linear in $x$, i.e.

\[ \forall (t, b) \in \mathbb{R} \times B \text{ the map } x \mapsto \varphi_t(b, x): \mathbb{R}^d \rightarrow \mathbb{R}^d \text{ is linear and } \]

\[ \Phi_t(b, x) = (\Theta_t b, \varphi_t(b, x)) \]

It is called measurable (continuous, differentiable), if $B$ is a measure space (topological space, differentiable manifold, resp.) and $\Phi$ is measurable (continuous, differentiable, resp.).

$B$ is called the base space.

\[ \Phi_{t+s}(b, x) = (\Theta_{t+s} b, \varphi_{t+s}(b, x)) \]

\[ \Theta_t(\varphi_s(b, x)) = \varphi_t(\Theta_s b, \varphi_s(b, x)) = (\Theta_t \Theta_s b, \varphi_t(b, \varphi_s(b, x))) \]

We get:

\[ \varphi_{t+s}(b, x) = \varphi_t(\Theta_s b, \varphi_s(b, x)) \quad t, s \in \mathbb{R} \]

**Ex.** Linear time-varying ODE: $\dot{x} = A(t)x$

Let $A: \mathbb{R} \rightarrow \text{gl}(d, \mathbb{R})$ be a uniformly continuous function.

Define $\Phi: \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ by $\Theta_t(c) = t + t$

and $\varphi: \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$

$\varphi(t, t, x) = x(t, t+1)x_0$
Where \( X \) is the fundamental solution, i.e., solving
\[
\dot{X}(t) = A(t)X(t), \quad X(\tau, \tau) = I \quad \text{and} \quad \Phi_t(\tau, x) = (\Theta_t \tau, \Psi_t(\tau, x))
\]
\[
\Psi_t(\tau, x) = X(t + s, t + s + \tau)x_0
\]
\[
\Phi_t(\Theta_t \tau, \Psi_t(\tau, x)) = X(t + s, t + s + \tau)X(s, t + \tau)x_0
\]

**Ex** Let \((\Omega, \mathcal{F}, P)\) be a probability space:

- \(\mathcal{F}\) - \(\sigma\)-algebra on \(\Omega\),
- \(P\) - probability measure.

Let \(\Theta : \mathbb{R} \times \Omega \to \Omega\) be a measurable map with
\[
\Theta_{t+s} = \Theta_t \cdot \Theta_s, \quad t, s \in \mathbb{R}, \quad \Theta_0 = \text{id}_\Omega
\]
and
\[
P(A) = P(\Theta_t A) \quad \text{for all} \quad A \in \mathcal{F}, \quad t \in \mathbb{R}.
\]

(\(P\) is \(\Theta\)-invariant)

Linear random dynamical system: A linear skew product flow
\[
\Phi : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \Omega \times \mathbb{R}^d \quad \text{with} \quad \Phi(\Theta, \varphi),
\]
with \(\Theta\) as above and \(\varphi : \Omega \times \mathbb{R}^d \to \mathbb{R}^d\) measurable.

**Ex** Robust Linear Systems
\[
\dot{x}(t) = A(u(t))x(t) = \left[A_0 + \sum_{i=1}^{m} u_i(t)A_i\right]x(t)
\]
where \(A_0, A_1, \ldots, A_m\) are given \(d \times d\) matrices,
\[
u = (u_i)_{i=1, \ldots, m} \in U = \{\text{ functions } u : \mathbb{R} \to \mathbb{R}^d \text{ integrable on every } [t, t+\frac{1}{2}] \text{ bounded interval}\}
\]
for example,
\[
\dot{x}_1 = x_2
\]
\[
\dot{x}_2 = -(1 + u(t))x_2(t)
\]
\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} + u(t)
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

Base space: \(B = U\)
\[
(\Theta_t u)(s) = u(t + s), \quad s \in \mathbb{R}
\]
\[
(\Theta_{t+s} u)(s) = u(t + s + \tau) = (\Theta_{\tau} u)(t + s)
\]
\[
= (\Theta_t \Theta_{\tau} u)(s)
\]
\( \psi_t(u, x_0) \) — solution of \( \dot{x} = A(u(t))x \), \( x(0) = x_0 \) at time \( t \).

\( \psi_{t+s}(u, x_0) = \psi_t(\theta_s u, \psi_s(u, x_0)) \)

\( \Phi : \mathbb{R} \times U \times \mathbb{R}^d \rightarrow U \times \mathbb{R}^d \)

Ex. Consider \( y = f(y) \) in \( \mathbb{R}^d \) with \( f \in C^1 \) and suppose \( B \subset \mathbb{R}^d \) is invariant under the corresponding flow \( \theta_t \).

\( \Theta_t y_0 \) is the solution of this ODE with 
\( \Theta_0 y_0 = y_0 \) (initial condition \( y(0) = y_0 \))

Denote the Jacobian of \( f \) in \( \Theta_t(y_0) \) by

\( Df(\Theta_t y_0) \)

Then consider \( \dot{y}(t) = f(y(t)) \) \( \begin{cases} \dot{y}(t) = Df(y(t))y(t) & \text{in } \mathbb{R}^d \times \mathbb{R}^d, y(0) = y_0, x(0) = x_0 \end{cases} \)

Second component \( \dot{x}(t) = Df(\Theta_t y_0) x(t) \)

\( \Phi_t : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d \)

\( (y_0, x_0) \mapsto (\Theta_t y_0, \psi(t, y_0, x_0)) \) is linear

If \( B \) is compact, then we have topological dynamics.

If \( B \) is a probability space \( \rightarrow \) smooth ergodic theory.

D. Ruelle, R. Mañé

We will start with
\( \dot{x} = A(t)x(t) \) \( t \mapsto A(t) \) periodic.