A \ d \times \ d \ matrix
\dot{x} = Ax, \ x(0) = x_0.
x_0 \in L(\chi_j) \iff x(t, x_0) \ has \ Lyapunov \ exponent \ \chi_j
\text{i.e. lim}_{t \to \pm \infty} \frac{1}{t} \ln \|x(t, x_0)\| = \chi_j \quad \mathbb{R}^d \rightarrow \mathbb{R}^d \ linear \ isomorphism
\Phi_t \in \mathbb{R}^d \rightarrow \mathbb{R}^d
\Phi_t^A \in \mathbb{R}^d \rightarrow \mathbb{R}^d
\Phi_t^B \in \mathbb{R}^d \rightarrow \mathbb{R}^d
\Phi_t = e^{At} : \mathbb{R}^d \rightarrow \mathbb{R}^d
\text{x(t) = A(t) x(t) - more complicated case, we will study it tomorrow,}

Consider \ \Phi_t = e^{At}
\text{it induces a flow on}
\text{projective space:}
P^{d-1} = \{ \text{lines through 0} \}

4. Chain Recurrence and Morse Decompositions.
Let \ \Phi: \mathbb{R} \times M \rightarrow M \ - \ a \ flow \ on \ a \ compact \ metric \ space \ M.
[ \Phi_0 = Id, \ \Phi_{t+s} = \Phi_t \circ \Phi_s ]
Def. For \ x \in M \ the \ omega \ limit \ set:
\omega(x) = \{ y \in M \mid \text{there is } t_k \to \infty \ \text{with } \Phi_{t_k} x \to y \}
and the \ alpha \ limit \ set:
\alpha(x) = \{ y \in M \mid \text{there is } t_k \to -\infty \ \text{with } \Phi_{t_k} x \to y \}
Ex. \ M \subset \mathbb{R} \ compact:
\omega(x), \alpha(x) \ are \ one-point \ sets
\text{a solution cannot go back and forth between } y \ \text{and } z, \ \text{because of the uniqueness of the solution}
In $\mathbb{R}^2$:

**Poincaré–Bendixon**

If $w(x)$ does not contain an equilibrium, then $w(x)$ is a periodic solution.

In general:

there are continua of equilibria

Needed: a notion which is more general than $w$-limit sets $x = 0$

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R. Bowen, C. Conley.

Let $\varepsilon, T > 0$ and $x, y \in M$. An $(\varepsilon, T)$-chain from $x$ to $y$ is given by times $T_0, \ldots, T_{n-1} \geq T$, points $x_0 = x, x_1, \ldots, x_n = y$ with

\[ d(\Phi_{T_i}(x_i), x_{i+1}) < \varepsilon \text{ for } i = 0, 1, \ldots, n-1. \]

A set $K \subset M$ is chain transitive if for all $x, y \in K$ and all $\varepsilon, T > 0$ there is an $(\varepsilon, T)$-chain from $x$ to $y$. The chain recurrent set $R$ is $R = \{ x \in M |$ for all $\varepsilon, T > 0$ there is an $(\varepsilon, T)$-chain from $x$ to $x \}$

**Ex.**

\[
\begin{array}{ccc}
0 & \xrightarrow{\varepsilon} & 1 \\
\text{equilib.} & & \text{equilib.}
\end{array}
\]

there is an $(\varepsilon, T)$-chain from 0 to 1, but not from 1 to 0.

\[ R = \{ 0, 1 \} \]

The connected components of $R$ are called chain recurrent components.
Fact: \( \Phi \) restricted to a chain recurrent component is chain transitive.

An \((E_1, T)\) chain is also an \((E_2, T)\) chain for \(E_2 \geq E_1\).

If \(T_1 < T_1^*\) then an \((E, T_1)\) chain is also an \((E, T_1^*)\) chain.

\( (T_1 > T_1^* > T_2)\)

Prop. A set \(K \subset M\) is chain transitive iff for all \(x, y \in K\) there is an \((E, 1)\)-chain with jump times \(T_i \in [1, 2]\).

**Proof**

\[
\begin{array}{c}
\Phi_{T_i}(x) \rightarrow y \\
x \quad \Phi_{T_i}(x) \quad T_i \geq T
\end{array}
\]

Jump times may be chosen in \([1, 2]\).

Suppose we have jump times in \([1, 2]\). Can we make them bigger?

\[
\begin{array}{c}
\Phi_t(x) \rightarrow y \\
x \quad \Phi_t(x)
\end{array}
\]

By uniform continuity, we can choose an \(\varepsilon\) and make the jump time bigger.

**Def** A compact set \(K \subset M\) is isolated invariant, if \(\Phi_t(x) \in K\) for all \(x \in K\), \(t \in \mathbb{R}\) and there exists a neighborhood \(N\) of \(K\) (i.e., \(K \subset \text{int}N\)), such that \(\Phi_t(x) \in N\) for all \(t \in \mathbb{R}\) implies \(x \in K\).

**Def** A Morse decomposition for a flow \(\Phi\) on \(M\) is \(\{M_1, \ldots, M_k\}, M_i \subset M\) of nonvoid, pairwise disjoint and isolated invariant sets, such that

(i) for all \(x \in M\), \(x \in \bigcup_i M_i\);

(ii) suppose there are \(M_j, M_i, \ldots, M_k\) and \(x_1, \ldots, x_k \in M\) with the property \(x_i \subset M_{i+1}\) and \(x_i \subset M_{j_i}\) for \(i = 1, \ldots, k\), then \(M_j \neq M_k\) (no cycles).
Example \( M = [0, 2] \)

The flow \( \Phi \) has equilibria
\[ 0, 1, 2; \text{ and for } x \in (0, 1) : \omega(x) = \{1\}, \alpha(x) = \{0\} \]
\[ \text{and for } x \in (1, 2) : \omega(x) = \{2\}, \alpha(x) = \{1\} \]

Morse decomposition: \( M_1 = \{0\}, M_2 = [1, 2]; M_3 = \{2\} \)
\[ M_4 = \{0\}, M_5 = \{1\}, M_6 = \{2\} \]
\[ M_7 = [0, 2] \]

A Morse decomposition \( \{M_1, \ldots, M_n\} \) is finer than \( \{M'_1, \ldots, M'_n\} \)
if for all \( M'_i \) there is \( M_j \) with \( M_j \subset M'_i \).

Thm (Conley) There exists a finest Morse decomposition
iff the chain recurrence set \( R \) has only finitely many
connected components.

In this case, these Morse sets are the connected components
of \( R \), i.e. they are the chain recurrent components.

Prop Let \( \Phi, \Psi \) be continuous dynamical systems on
a compact metric space \( M \) and suppose there is a
topological conjugacy (\( C^0 \)) \( h \), i.e.

\[
\begin{align*}
M & \xrightarrow{\Phi} M \\
& \downarrow h \\
M & \xrightarrow{\Psi} M \\
\end{align*}
\]

Then for all \( x \in M \) \( h(\omega(x)) = \omega(h(x)) \), \( h(\alpha(x)) = \alpha(h(x)). \)

\textbf{Proof:} \( y \in \omega(x) \) iff \( \exists t_k \to \infty : \Phi_{t_k}(x) \to y. \)

Then \( h(\Phi_{t_k}(x)) = \Psi_{t_k}(h(x)) \)

\[ h \left( \begin{array}{c} y \end{array} \right) \]

Thus \( \Psi_{t_k}(h(x)) \to h(y) \). We have shown: \( y \in \omega(x) \Rightarrow h(y) \in \omega(h(x)) \)

Using \( h^{-1} \) we get the other inclusion. \( \quad \square \)

Corollary If \( \Phi \) and \( \Psi \) are top. conjugate flows, then a
Morse decomposition for \( \Phi \) is mapped onto a Morse
decomposition for \( \Psi \). A finest Morse decomposition
is mapped onto a finest Morse decomposition.
\textbf{5. Flows on Projective Space.}

\[ \dot{x} = Ax \quad \Phi_t x = e^{At} x, \quad t \in \mathbb{R} \text{ in } \mathbb{R}^d \]

Projective space \( P^{d-1} = \{\text{lines through the origin}\} \)

\[ S^{d-1} = \{x \in \mathbb{R}^d \mid \|x\| = \sqrt{x_1^2 + \ldots + x_d^2} = \|x\| \} \quad \text{Euclidean scalar product} \]

\( P^{d-1} \) can be identified with the unit sphere, when the opposite points are identified. The projection from \( \mathbb{R}^d \) is \( P : \mathbb{R}^d \setminus \{0\} \rightarrow P^{d-1} : x \mapsto \frac{x}{\|x\|} \)

Define the distance:

\[ d(P_x, P_y) = \min \{ \| \frac{x}{\|x\|} - \frac{y}{\|y\|} \|, \| \frac{-x}{\|x\|} - \frac{-y}{\|y\|} \| \} \]

This makes \( P^{d-1} \) into a compact metric space. \( \Phi_t = e^{At} \) induces a flow on \( P^{d-1} \), denoted by \( P\Phi_t \)

\textbf{Exercise:} \( s(t) = \frac{x(t)}{\|x(t)\|} \)

Write down a differential equation for \( s(t) \).

\[ A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ \text{equilibrium} \]

Suppose \( m \in \sigma(A) \cap \mathbb{R} \) and \( x \in \mathbb{R}^d \) is a corresponding eigenvector of \( A \). Then \( e^{At} x = e^{At} x \), hence \( P_x \) is an equilibrium for \( P\Phi_t \)

\[ P\Phi_t x \]

\[ = P\Phi_t (P_x) \]
More general: If $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$, then

$e^{At} = e^{\lambda t} e^{At}$ - it does not change the picture on the unit sphere.

$A = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ ($\lambda = 2$)

$e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

$e^{At} x_0 = \begin{pmatrix} x_0 + t x_0 \\ x_0 \end{pmatrix}$

Eigenspace $(0,1)(1,0) = 0$

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Theorem: Let $P \Phi$ be the projected flow in $P^{d-1}$ for $\Phi_t = e^{At}$, $A \in \text{gl}(d, \mathbb{R})$. Then

(i) $P \Phi$ has $l$ chain recurrent components $\{M_1, \ldots, M_l\}$, where $l$ is the number of different Lyapunov exponents (i.e. of different real parts of eigenvalues) of $A$.

(ii) For each Lyapunov exponent $\lambda_i$ one $M_i = PL(\lambda_i)$, where $L(\lambda_i)$ is the Lyapunov space.

(iii) For the $M_i$ define $P^i M_i = \{x \in \mathbb{R}^d \mid \exists t, P x \in M_i \}$. Then $P^i M_i = L(\lambda_i)$ and $\mathbb{R}^d = P^1 M_1 \oplus \ldots \oplus P^i M_i$.

$P^T T^{-1} A T = J^R$ real Jordan form

$e^{jt} = e^{T^{-1} A T} = T^{-1} e^{AT} T \iff T e^{jt} = e^{AT} T$, i.e. $e^{jt}$ and $e^{AT}$ are
conjugate. Then also $P(e^{s_t})$ and $P(e^{A_t})$ are conjugate. It suffices to construct the finest Morse decomposition in $P^{d-1}$ for a Jordan matrix. Suppose W.l.o.g. 

$A = \text{blockdiag}(J_1, \ldots, J_k)$

$M \in \sigma(A) \setminus \mathbb{R}: \quad J_i = \begin{pmatrix} M^1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M \in \sigma(A) \setminus \mathbb{R} \quad J^i = \begin{pmatrix} \lambda & -\nu \\ \nu & \lambda \end{pmatrix}$

It suffices to look at generalised e. spaces for eigenvalues with real part $\lambda$.

Subtracting $A$ from the diagonal, we get the same projection in $P^{d-1}$ W.l.o.g. : real part = 0.

$L(A, 0) = \bigoplus_{\text{Re} \, \mu = 0} E(A, \mu)$

We have to show that $P(L(A, 0))$ is chain transitive.

Consider a Jordan block corresponding to eigenvalue 0.

$(0, 1), (0, 0), (0, 1, 0), (0, 0) \in \text{projective space solutions tend}$

to the projected eigenvector for $t \to \pm \infty$ $\to$ the corresponding projected subspace is chain transitive.

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

Look at:

$\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$\alpha, \beta \in \mathbb{C} \setminus \mathbb{R}$

is an eigenvector for the eigenvalue 0.

We get a continuum of equilibria on the unit sphere.

This implies that $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ projected to $P^4$ are in the same chain transitive set.

This implies that $P^4$ is a chain transitive set.

So, $PE(A, 0)$ is chain transitive.

Let $M = 0 + i \nu \in \sigma(A) \setminus \mathbb{R}$. $e^{(0 \ -\nu) t} = \begin{pmatrix} \cos \nu t & -\sin \nu t \\ \sin \nu t & \cos \nu t \end{pmatrix}$

Periodic solutions with period $\frac{2\pi}{\nu} \in \mathbb{R}^2$ all points on a periodic solution are chain transitive.
For higher dimension Jordan blocks:
\[
\begin{pmatrix}
0 & -2 & 1 & 0 \\
2 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 \\
0 & 0 & 2 & 0
\end{pmatrix}
\]
solutions in projective space are attracted to the projected eigenspace.

Jordan blocks corresponding to \(0 + iv_1\) are more complicated to analyze (given in the notes).