Jordan Canonical Form

In dynamical systems one considers:

\[ x(t+1) = A x(t) \]

or

\[ \dot{x} = A x \]

If \( P \) is an invertible matrix:

\[ P x(t+1) = P A P^{-1} P x(t) \]

\[ y(t+1) = B y(t) \]

We would like to make \( B \) as simple as possible.

**Definition:** \( B \) is similar to \( A \) provided there exists an invertible \( P \) with \( P A P^{-1} = B \).

This leads to Jordan Canonical Form.

Let \( A_{n \times n} \) be a complex matrix.

\( A \) is similar to a block-diagonal matrix

\[ P A P^{-1} = \begin{bmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{bmatrix} \]

where each of the blocks is a matrix

\[ J_m(\lambda) = \begin{bmatrix} \lambda & 1 & 0 \\ & \ddots & \ddots \\ & 0 & \lambda \end{bmatrix} \]

for some \( m \) and \( \lambda \).

The Jordan Canonical Form is very sensitive to perturbations.

Eq. \[ J_{JC}(A) \]

\[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

\[ \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \]

\[ \begin{bmatrix} 1 & 0 \\ 0 & i - \varepsilon \end{bmatrix} \]

\[ \begin{bmatrix} 1 & 0 \\ \varepsilon & 1 \end{bmatrix} \]

The diagonal entries of \( J_{JC}(A) \) are the eigenvalues of \( A \).

If we start with a real matrix \( A \) and insist that \( P \) be real, then \( A \) is similar to a block-diagonal matrix, each of whose blocks is either \( J_m(\lambda), \lambda \in \mathbb{R} \), some \( m \), or

\[ J_{2m}(a+bi) = \begin{bmatrix} a & b & 1 & 0 \\ -b & a & 0 & 1 \\ & & \ddots & \end{bmatrix} \]

for some \( a+bi \), \( b \neq 0 \), and some \( m \).
**Nonnegative Matrices.**

$A \geq 0$ means that each entry of $A$ is nonnegative real.

$A > 0$ means that each entry of $A$ is positive.

A square $A$ is called **positive** if $\rho(A) = \max \{ \lambda \}$ over all $\lambda$ an **e-value** of $A$.

What can you say about the e-values of $A > 0$?

**Perron's Thm**

Let $A_{n \times n}$ be a positive matrix. Then

(a) $\rho(A)$ is an eigenvalue of $A$.

(b) $f$ is a positive e-vector of $A$ corresponding to $\rho(A)$.

(c) If $\lambda$ is an e-value of $A$ and $\lambda < \rho(A)$, then $\lambda < 1$.

In complex plane:

![Complex Plane Diagram](image)

**Brouwer's Fixed Point Thm**

If $f : D^n \rightarrow D^n$ is a continuous function on disk in $\mathbb{R}^n$, then $f$ has a fixed point (i.e. $f(x) = x$).

Case $n = 1$: $f : [0, 1] \rightarrow [0, 1]$.

**Proof of Perron's Thm**

Let $A_{n \times n} > 0$.

Let $S = \{ x \in \mathbb{R}^n : x \geq 0, x_1 + x_2 + \cdots + x_n = 1 \}$.

Claim: $S$ is topol. equiv. to $D^{n-1}$.

Consider $f : S \rightarrow S$ by $f(x) = \frac{Ax}{\|Ax\|}$.

Note $A > 0$, $x > 0$, so $\|Ax\| > 0$ implies $\|Ax\| > 0$.

Clearly $f$ is continuous.

By the Brouwer's Fixed Point Thm, there is an $x$ such that $f(x) = x$.

I.e. $Ax = (\|Ax\|)x$.

So $x$ is an e-vector with corresponding e-value $\|Ax\|$.

Let's show that $\|Ax\|$ is $\rho(A)$.

Let $\lambda$ be an e-value of $A$ and $y^*$ a left e-vector.

So $y^*A = \lambda y^* \Rightarrow |y^*A| = |\lambda y^*| = |\lambda| |y^*|$.

Using the entrywise modulus, $|y^*A| \geq \Delta$-inequality.
the entry of \( y^T A \) is \( \sum \frac{y_i a_{ij}}{y_j a_{ij}} \) for \( i \neq j \). Thus \( y^T A \geq 1_{k_1} 1_{y^T} \) and \( y^T A x \geq \lambda x^T y^T \) for all \( x \).

So \( A^T x \geq \lambda x \). Thus \( \rho(A) = \lambda^T A x \)

\( x \) must be positive; \( A x = \frac{(A^T A x) x}{\sum \lambda_i} \).

To prove (c) we need to show that if \( \lambda = \rho(A) \), then

\( A = \rho(A) \).

Assume \( \lambda = \rho(A) \), i.e. \( \lambda = \rho(A) e^{i\theta} \) for some \( \theta \).

This requires equalities throughout our analysis.

We need equality in our application of the triangle inequality

\( \sum \frac{y_i a_{ij}}{y_j a_{ij}} \leq \sum \frac{y_i a_{ij}}{y_j a_{ij}} \).

This requires that the arguments of \( y_i a_{ij} \) are the same. Since all \( a_{ij} \) are positive, this means that the arguments of \( y_i \) are the same.

So \( \lambda = e^{i\theta} \) for some \( \theta \).

\( \lambda \) is a left eigenvector of \( A \), i.e. \( \lambda y^T A = \lambda y^T \).

By assumption, \( \lambda = \rho(A) \), so \( \lambda = \rho(A) \).

How much of Perron's Theorem carries over to nonnegative matrices?

Examples

1. \( A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \)

   - values: \( 2, 1 \)
   - \( \rho(A) \) is a real e-value
   - vector corresponding to \( \rho(A) \) is \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) not positive.

2. \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \)

   - values: \(-1, 1\)
   - \( \rho(A) = 1 \) is an e-value
   - But \( \lambda = 1 \) satisfies \( \lambda = 1 \).
Where did we use positivity in our proof?

A > 0.
Consider \( f : S \to S \) by \( f(x) = \frac{Ax}{\|Ax\|} \).

How can we guarantee that \( \|Ax\| \) is nonzero for all \( x \in S \).
\( \|Ax\| = \text{sum of the entries of } Ax \). Answer: no column of \( A \) is all 0's.

\( A \left[ \begin{array}{c} x \\ \hat{y} \\ \delta \\ \gamma \\ \beta \\ 0 \end{array} \right] = 0 \) iff 1st \( k \) columns of \( A \) are 0's.

by the Brouwer F.T. \( f \) has a fixed point.
J.e. there is an \( x \in S \) with \( Ax = (\|Ax\|)x \)

We'd like \( x > 0 \). How can we guarantee this?

\[
A \left[ \begin{array}{c} x \\ \hat{y} \\ \delta \\ \gamma \\ \beta \\ 0 \end{array} \right] = 0 
\]

Ans. By requiring that there is no permutation matrix \( P \)
such that \( PAP^T = k \left[ \begin{array}{c} \text{orthogonal} \\ \text{non-} \| \text{orthogonal} \end{array} \right] \)

\( \det A \) is irreducible if there does not exist a permutation matrix \( P \) such that \( PAP^T = k \left[ \begin{array}{c} \text{orthogonal} \\ \text{non-} \| \text{orthogonal} \end{array} \right] \)

Let's assume that \( A \geq 0 \) and \( A \) is irreducible.
(i.e. \( n \geq 2 \), then \( A \) irreducible \( \Rightarrow A \) has no col. of all 0's).

We know that \( Ax = (\|Ax\|)x \) and \( x > 0 \).
Let \( y^* \) be a left eigen-vector for \( A \) corresponding to \( \lambda \).

\[
y^*A = \lambda y^* \\
1y^*\|A\| = 1x^*y^* \|A\| \\
1y^*\|A\| \geq 1x^*y^* \|A\| \xrightarrow{\|A\|} (\|Ax\|) \|y\| \|x\| \\
\]

Cancelling gives: \( \|Ax\| \geq 1x^* \) i.e. \( \rho(A) = \|Ax\| \)
Theorem (Perron–Frobenius)

Let $A_{n \times n} \geq 0$ with $n \geq 2$ and $A$ irreducible. Then

(a) $\rho(A)$ is an e-value of $A$

(b) $I$ an eigen-vector $x$ of $A$ corresponding to $\rho(A)$ with $x > 0$.

Note: $[0 \ 1]^{T}$ irreducible, $A \geq 0$, but e-values are $-1, 1$.

Irreducibility is not enough to show $1\lambda_1 = \rho(A) \Rightarrow \lambda = \rho(A)$

We'd still like to say something about e-values $\lambda$ of $A \geq 0$ irreducible with $1\lambda_1 = \rho(A)$.

This comes down to analyzing equality in $\Delta$-inequality.

$y^{*} A = \lambda y^{*}$

$|y^{*} A| \geq |\lambda_1| |y^{*}|$

If $1\lambda_1 = \rho(A)$, then we must have equality in this $\Delta$-inequalities.

The $j^{th}$ inequality

$|\overline{y}_{1} a_{1j} + \overline{y}_{2} a_{2j} + \ldots + \overline{y}_{n} a_{nj}| \leq |y_{1} a_{1j} + \ldots + y_{n} a_{nj}|$

Equality iff all nonzero terms among $\overline{y}_{k} a_{kj}$ have the same argument.

iff all the $y_{k}$ with $a_{kj} \neq 0$ have the same argument.

In terms of the digraph of $A$:

Equality requires $\arg(y_{k}) = \arg(y_{l})$ whenever $k \rightarrow l$.

Let's look at $y^{*} A = \lambda y^{*} = e^{i\theta} \rho(A) y^{*}$

The $j^{th}$ entry $\sum_{k \rightarrow j} \overline{y}_{k} a_{kj} = e^{i\theta} \rho(A) \overline{y}_{j}$

All $\overline{y}_{k}$ where $k \rightarrow j$ have the same argument $\beta$.

$\sum_{k \rightarrow j} \overline{y}_{k} a_{kj}$ has argument $\beta$

$e^{i\theta} \rho(A) \overline{y}_{j}$ has argument $\theta + \arg(\overline{y}_{j})$ \Rightarrow $\beta = \theta - \arg(\overline{y}_{j}) \mod(2\pi)$

$\arg(y_{k}) + \theta = \arg(y_{j})$
Upto: \( k - j \)

\[
\arg(y_k) + \theta = \arg(y_j)
\]

So: \( \theta = \frac{2\pi}{p} \) for some \( p \geq 1 \)

and

\[
PAP^T = \begin{bmatrix}
A_1 & 0 \\
0 & A_2 \\
& \\
A_k
\end{bmatrix}
\]

if \( p \geq 2 \)

or \( p = 1 \), previous proof shows \( \lambda = p(A) \).

**Defn:** \( A_{n \times n} \geq 0 \) is primitive provided the \( \gcd \) of the cycle lengths of the cycles in \( D(A) \) is \( 1 \).

Can show \( A \) is primitive iff \( A \) is irreducible and \( A^m > 0 \) for some \( m \).

**Ex:**

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\]

\( D(A) \) has 3 cycles

\( \lambda \) is not primitive

\( 3 \) - primitive.

Can't be the spectrum of an irreducible nonnegative matrix.
Thm (Perron–Frobenius)
\( A > 0 \), irreducible and primitive.

Then
- \( \rho(A) \) is an e-value of \( A \)
- \( \exists \) a corresponding e-vector \( x \) with \( x > 0 \)
- \( \lambda \) an e-value of \( A \) and \( 1 \lambda \rho(A) \Rightarrow \lambda = \rho(A) \).

Ranking players in round robin tournaments.

A tournament on \( n \) players consists of a competition where each pair of players play exactly one game against each other and no ties are allowed. Total of \( \binom{n}{2} \) games.

We can record the results of a tournament by a matrix
\[
A = [a_{ij}] \quad a_{ij} = \begin{cases} 
1 & \text{if } i \to j \\
0 & \text{otherwise} 
\end{cases}
\]
or
by a digraph

\[
\text{Eq 1. } A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}
\]

Results
- 1 beats 2
- 3 beats 1
- 3 beats 2
- 4 beats 1, 2, 3

Sometimes we only show the down arcs

\[
\text{Eq 2. } A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\]

4 is the best player

How can we rank or determine the strengths of the players?

\[ A + A^T = \begin{bmatrix} 0 & 1's \\ 1's & \vdots \\ \vdots & \vdots \\ 1's & 0 \end{bmatrix} = J - I \]

\( S = A \| \) is the vector that records the number of wins for each player.

Challenge: Find a way to rank the players of the tournament.
\[
E_x = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}
\]
\[
A = \begin{bmatrix}
2 &  &  &  &  & \\
2 &  &  &  &  & \\
3 &  &  &  &  & \\
\end{bmatrix}
\]

Player 4 wins 3 games: beat 1, 5, 6
Player 5 wins 3 games: beat 1, 2, 6
Player 6 wins 3 games: beat 1, 2, 3

Player 4 could argue they beat better players, so they are the best.
So S only counts wins; it does not account for who
those wins were against.

\[
(A S)_i = \sum_j s_{ij} = \text{sum of the scores of the players that } i \text{ beats}
\]

\[
\begin{bmatrix}
1 \\
5 \\
6 \\
8 \\
7 \\
6
\end{bmatrix}
\]

Any time we have a "strength" vector \( v \),
\[
\frac{A v}{||A v||}
\]

is arguably a "better" strength vector because it
incorporates the strengths of the players a player beats.

\underline{Scheme (1950's Kendall-Wei')}

Start with \( v_0 = 1/n \)

\[
v_{k+1} = \frac{A v_k}{||A v_k||}
\]

Note: \( ||v_k|| = 1 \) for all \( k \)

If \( A \) is irreducible, then \( \lim_{k \to \infty} v_k = x \) is an eigen vector

corresponding to \( \rho(A) \)

K-W proposed ranking the players according to \( x \),

i.e. \( i \) is stronger than \( j \) whenever \( x_i > x_j \).
\[ P = 1.7194 \]

\[ X = \begin{bmatrix} 16 \\ 18 \\ 10 \\ 20 \\ 26 \end{bmatrix} \]

One can measure how competitive the tournament is by

\[ \text{var}(X) = \frac{\sum_{i \neq j} (X_i - X_j)^2}{(X^T X)} \]

Larger variance corresponds to less competitive.

\[
\text{Var}(x) = \frac{\sum_{i \neq j} (x_i - x_j)^2}{x^T x} = \left( \frac{\sum_{i=1}^n (n-1)x_i^2 - 2 \sum_{i \neq j} x_i x_j}{\sum_{i=1}^n x_i^2} \right) = \\
= \frac{\sum_{i=1}^n (n-1)x_i^2 - x^T (J-I)x}{\sum_{i=1}^n x_i^2} = \frac{\sum_{i=1}^n (n-1)x_i^2 - x^T (A + A^T)x}{\sum_{i=1}^n x_i^2} = \\
= \frac{\sum_{i=1}^n (n-1)x_i^2 - 2p(A)x^T x}{\sum_{i=1}^n x_i^2} = \left[ n-1 - 2p(A) \right]
\]

So \( \text{Var}(X) = n-1 - 2p(A) \)

**Uphalt:** \( p(A) \leq \frac{n-1}{2} \) and \( \text{Var}(X) \) is large when \( p(A) \) is small.

Q: (Brualdi - Li)

Which tournaments on \( n \) players are the most competitive? i.e. which tournaments on \( n \) players have the largest \( p(A) \)?

\[ \max p(A) \]

A: \( A(0,1) \)

\[ A + A^T = J - I. \]
\[
\frac{n = 3}{1 \hspace{1cm} \text{or} \hspace{1cm} 1 \hspace{1cm} 2 \hspace{1cm} 3}
\]
\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]
\[
\rho = 1 \quad \rho = 0
\]

\[
\frac{n = 5}{1 \hspace{1cm} 2 \hspace{1cm} 3 \hspace{1cm} 4 \hspace{1cm} 5}
\]
\[
\begin{bmatrix}
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0
\end{bmatrix}
\]
\[
A_1 = 24 \quad \rho = 2
\]

Best possible.

If \( n = 2k + 1 \) any regular tournament (i.e. each player beats \( k \) players) has \( \rho = k = \frac{n - 1}{2} \).

So the regular tournaments are most competitive (for \( n \) odd).

For \( n \) even there are no regular tournaments.

(Why? \( \# \text{games} = \frac{n(n-1)}{2} \), \( \# \text{of games won by each player in reg. turn} = \frac{n-1}{2} \) odd if \( n \) is even.

Brauidd - Li

\[\text{What is } \max_{n \times n \text{ matrix}} \rho(A) ? \]

\[\text{Conjecture: occurs for} \quad n \text{ known for } n \leq 12.\]