Laplacians

For a graph $G$, the Laplacian $L$ is defined as:

$$L = \begin{bmatrix} b_{ij} \end{bmatrix}$$

where

- $b_{ij} = \frac{\deg(i)}{i=j}$ if $i\neq j$ and $i,j$ are adjacent in $G$,
- $b_{ij} = 0$ if $i\neq j$ and $i,j$ are not adjacent in $G$.

**Matrix-tree theorem**

The adjoint of $L$, denoted $\text{adj} L = c(G) J$, where $c(G)$ is the number of spanning trees of $G$.

**Proof:**

Let $B = \begin{bmatrix} e_1 e_2 \ldots e_m \end{bmatrix}^T$ be the incidence matrix of $G$ with $m$ edges and $n$ vertices.

Let $\tilde{G}$ be an arbitrary orientation of $G$.

$$\tilde{G} = \begin{array}{c}
\begin{array}{c}
1 \quad 3 \quad 4 \\
2
\end{array}
\end{array}$$

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\begin{array}{c}
1 \quad 3 \quad 4 \\
2
\end{array}
\end{array}$$

$$B = \begin{bmatrix}
-1 & 1 & 0 & 0 \\
1 & 0 & -1 & 1 \\
0 & -1 & 1 & -1 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

**Note:**

$$(BB^T)_{ii} = \deg(i)$$

$$(BB^T)_{ij} = \begin{cases} -1 & \text{if } i \neq j \\
0 & \text{otherwise} \end{cases}$$

So $BB^T = L$.

$$L = n \begin{bmatrix} B & B^T \end{bmatrix}
$$

By Cauchy-Binet:

$$\det L = \sum_{S \subseteq \{1,2,\ldots,m\}, |S| = n-1} \det B[S,\bar{S}] \det B^T[S,\bar{S}]$$

$$= \sum_{S \subseteq \{1,2,\ldots,m\}, |S| = n-1} \left( \det B[S,\bar{S}] \right)^2$$
Take $S \subseteq \{1, 2, \ldots, m\}$ with $|S| = n - 1$

Let $G_S$ be the graph with edge $e_s$, $s \in S$

$G_S$ has $n$ vertices and $n - 1$ edges.

Either $G_S$ is not connected, or $G_S$ is a tree.

We will show that if $G_S$ is not connected, then

$\det B[\overline{n}, S] = 0$

and

if $G_S$ is a tree, then

$\det B[\overline{n}, S] = \pm 1$.

It follows that $\det L[\overline{n}, \overline{n}] = \#$ of spanning trees of $G$.

Let's see with an example what happens when $G_S$ is a spanning tree.

$S = \{1, 3, 4\}$ in example:

$B[\overline{n}, S] = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

after rearranging rows and columns we get:

$\begin{bmatrix} \pm 1 & 0 & 0 \\ \pm 1 & 0 & 0 \\ \pm 1 & 0 & 0 \end{bmatrix} \leftarrow$ pendant vertex (vertex of degree 1)

Now let's consider $G_S$ disconnected.

$B[\overline{n}, S] \not\sim \begin{bmatrix} 0 & \vdots \\ \vdots & \ddots & 0 \\ 0 & \vdots \end{bmatrix}_{n \times n - 1}$

rearranging rows and columns

If we delete one of the top rows:

If we delete one of the bottom rows:

$x$ has linearly dependent columns,

so the matrix has 0 determinant.

Each column of $x$ has a 1 and $a-1$.

i.e. the column sums of $x$ are all 0.

so $x$ and $B[\overline{n}, S]$ is not invertible, so $\det B[\overline{n}, S] = 0$. 
So we have shown that
\[ \det L[\overline{m}, \overline{n}] = C(G). \]

Now
\[ \Pi L \Pi^T = 0 \]
\[ \Pi L \Pi^T = 0 \]

So \( \text{adj} L \) is a mult. of \( \Pi L \Pi^T = J \).
Since \( \det L[\overline{m}, \overline{n}] = C(G) \), \( (\text{adj} L)_{\overline{n}, \overline{n}} = C(G) \).
So \( \text{adj} L = C(G)J \).

Let's investigate the spectral properties of \( L \).
1. \( L \) is symmetric, so \( L \)'s eigenvalues are real.
   - List as \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \)
2. \( L = BB^T \Rightarrow \) each \( \lambda \)-value is nonnegative
   - Why? Suppose \( Lx = \lambda x \), so \( BB^T x = \lambda x \)
     \[ x^T B B^T x = \lambda x^T x \]
     \[ (B^T x)^* (B^T x) = \lambda x^T x \]
     \[ \text{Squared length of } B^T x \]

So \( \lambda \geq 0 \).

3. \( LL^T = 0 = 0^T L \) so \( \lambda_1 = 0 \).

4. \( \lambda_2 \) has many special properties and it is called
   the algebraic connectivity of \( G \).
   - Thm (Fiedler) \( (n \geq 2) \)
   - (a) \( 0 \leq \lambda_2 \leq n \)
   - (b) \( \lambda_2 = 0 \) iff \( G \) is not connected
   - (c) \( \lambda_2 = n \) iff \( G \) is the complete graph
   - (d) \( \lambda_2 \) is a monotone increasing function on edges,
     i.e., \( \lambda_2 (G \cup e) \geq \lambda_2 (G) \) (adding an edge does not decrease the connectivity).
   - (e) \( \lambda_2 (G) - 1 \leq \lambda_2 (G - v) \) - deleting a vertex
   - (f) If \( G \) is not \( K_n \), then \( \lambda_2 (G) \leq \text{vertex connectivity} \)
     \( \leq \text{edge connectivity} \) i.e., min # of edges needed to disconnect the graph.
     vertex connectivity is the min number of vertices whose removal disconnects the graph.
Proofs of (a)-(c):

Note \( x^T L x = \sum_{i,j} d_{ij} x_i x_j = \sum_{i,j} \frac{1}{2} (x_i - x_j)^2 \) with \( i,j \in G \) and \( i \neq j \).

(a): Clearly, \( x^T L x \geq 0 \) for all \( x \).

So if \( x \) is an e-vector for \( \lambda_2 \), then

\[
\begin{align*}
x^T L x & = 0 \\
x^T (\lambda_2 x) & = \lambda_2 x^T x \\
\lambda_2^2 x^T x & \implies \lambda_2 \geq 0
\end{align*}
\]

\[\sum_{i=1}^{n} d_i = \text{tr}(L) = \lambda_1 + \lambda_2 + \ldots + \lambda_n = 0 + \lambda_2 + \ldots + \lambda_n > (n-1) \lambda_2\]

So \( n > \lambda_2 \).

Equality: \( d_i = n-1 \) for all \( i \), so \( \lambda_2 = n \) implies \( G = K_n \).

(c)

(b):

\( \Leftarrow \) Assume \( G \) is not connected.

Then \( L = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \), and \( L \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \),

So \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) are linearly independent e-vectors of \( L \) corresponding to \( \lambda = 0 \).

So \( \lambda_2 = 0 \).

\( \Rightarrow \) Assume \( \lambda_2 = 0 \).

Then there exists an e-vector \( x \) corresponding to \( \lambda = 0 \), which is not a multiple of \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \).

Note \( x^T L x = \sum_{i,j} (x_i - x_j)^2 \) for \( i \neq j \).

So \( x_i = x_j \) whenever \( i \neq j \).

If there is a path from \( i \) to \( j \) in \( G \), then \( x_i = x_j \).

Since \( x \) is not a multiple of \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), there exist \( i \) and \( j \) with \( x_i \neq x_j \). So there is no path from \( x_i \) to \( x_j \).

So \( G \) is not connected.
(d) Show: $\lambda_2(G) \leq \lambda_2(G_{\text{Max}})$.

$$\lambda_2(G) = \min_{\substack{x \in \mathbb{R}^n \mid x^T d = 0 \quad x^T x = 1}} x^T L x = \min \sum_{i \neq j} (x_i - x_j)^2$$

Courant-Fischer

$$\lambda_2(G_{\text{Max}}) = \min_{\substack{x \in \mathbb{R}^n \mid x^T d = 0 \quad x^T x = 1 \quad x \in G}} \sum_{i \neq j} (x_i - x_j)^2$$

We can assume by (d) that $v$ has degree $n-1$. WLOG $v = 1$.

By the Courant-Fischer:

$$\lambda_2(G_{\text{Max}}) = \min x^T L x \leq \min x^T L x$$

where $x^T d = 0$ and $x^T x = 1$.

$$x = \begin{bmatrix} 0 \\ y \end{bmatrix}$$

$$L = \begin{bmatrix}
-1 & \cdots & -1 \\
-1 & \cdots & L(G \setminus v) + I \\
\vdots & \ddots & \vdots \\
-1 & \cdots & -1
\end{bmatrix}$$

$$x^T L x = \begin{bmatrix} 0 \\ y^T \end{bmatrix} L \begin{bmatrix} 0 \\ y \end{bmatrix} = y^T (L(G \setminus v) + I) y = y^T L(G \setminus v) y + y^T y = y^T L(G \setminus v) y + 1$$

So

$$\min x^T L x = \min y^T L(G \setminus v) y + 1 = \lambda_2(G_{\text{Max}}) + 1$$

Any $e$-vector of $L$ corresponding to $\lambda_2$ is called a Fiedler vector.
Let $f$ be a Fiedler vector of $G$.
Let $c > 0$
Let $G_c$ be the induced graph of $G$ whose vertices are those $i$ s.t. $f_i \leq c$.

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{bmatrix}
\]

\[
f = \begin{bmatrix}
0.5 \\
1 \\
-3 \\
-4 \\
0 \\
-1.3 \\
-1.1 \\
-0.5 \\
\end{bmatrix}
\]

\[
G_t:
\begin{bmatrix}
2 & 3 \\
1 & 6 \\
5 & 7 \\
8 & 9 \\
\end{bmatrix}
\]

\[
G_0:
\begin{bmatrix}
5 & 6 \\
1 & 8 \\
3 & 9 \\
\end{bmatrix}
\]

\underline{Fiedler's Thm}

If $G$ is connected, then $G_c$ is always connected.

What does this mean for a tree?

\[f_1 = 0 \quad \text{not allowed}\]

the Fiedler's vector has to respect the connectivity of the graph.

$G$ - connected, $f$ - fiedler vector for $L$, $c > 0$

$G_c$ - the graph induced by $i$ such that $f_i \leq c$

\underline{Fiedler's Thm : $G_c$ is always connected.}

Proof (case when $c > 0$)
By contradiction. Assume $G_c$ is not connected.

\[
L = \begin{bmatrix}
L_1 & 0 & L_{13} \\
0 & L_2 & L_{23} \\
L_{31} & L_{32} & L_3 \\
\end{bmatrix}
\]

vertices of $G_c$

\[
f = \begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
\end{bmatrix}
\]

Know:

\[f_1 \leq c \]
\[f_2 \leq c \]
\[f_3 > c \]

Let

\[
g = f - \mathbb{1} \]

\[
\begin{bmatrix}
g_1 \\
g_2 \\
g_3 \\
\end{bmatrix}
\]

\[g_i \leq 0\]
By the interlacing inequalities

either $\lambda_1(L_1) \geq \lambda_2$

or $\lambda_1(L_2) \geq \lambda_2$

WLOG assume $\lambda_1(L_1) \geq \lambda_2$

Note: $Lg = LF - cL1$

$= \lambda_2 f - 0$

So: $L_1 g_1 + L_13 g_3 = \lambda_2 f_1 = \lambda_2 (g_1 + c I)$ (*)

So: $L_1 g_1 - \lambda_2 g_1 = c \lambda_2 I - L_13 g_3$

Premultiply by $g_1^T$:

$$g_1^T (L_1 - \lambda_2 I) g_1 = c \frac{g_1^T g_1}{\lambda_2} \stackrel{\text{Rayley quotient}}{\geq} 0$$

nonnegative

(all terms on both sides are zero:

$c \lambda_2 g_1^T I = 0$

so $g_1 = 0$ (since $c > 0$, $\lambda > 0$)

$g_1^T L_13 g_3 = 0$

Look at (*) :

$L_13 g_3 = \frac{\lambda_2 c I}{0}$

so $L_13 g_3 = 0$

$L_{13} \begin{bmatrix} + \\ + \\ + \\ + \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow L_{13} = 0$

But $L_{13} = 0$ implies that $G$ is not connected.

Tanner in 1984 showed that $\lambda_2(G)$ gives a measurement of the difficulty of disconnecting the graph.

Given $G$, let $S$ be a subset of vertices of $G$

with $S \neq \emptyset$ and $S$ contains not all vertices of $G$.

Then $\partial S = \{ e : e$ is an edge with one vertex in $S$ and one vertex out of $S \}$

\[ S \xrightarrow{\partial S} S \]
Let $G$ be a connected graph. Then
\[
\frac{|\partial S|}{|S||S'|} \geq \frac{\lambda_2(G)}{n} \quad \text{for all } S.
\]

Proof:
Let $x = [x_1, \ldots, x_n]^T$ where
\[
x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}
\]
We know that \( \lambda_2(G) = \min \{ \frac{v^T L v}{\|v\|^2} : \|v\|^2 = 1 \} \)

Let $y = x - \frac{A^T x}{n} 1$
Then \( A^T y = A^T x - \frac{A^T x}{n} 1^T = A^T x - A^T x = 0 \).

Now
\[
y^T L y = \sum_{i-j} (y_i - y_j)^2 = \sum_{i-j \in S} (y_i - y_j)^2 = \sum_{i-j \in S} \left( 1 - \frac{|x_i|}{n} + \frac{|x_j|}{n} \right)^2 = |\partial S|
\]
\[
y_i = \begin{cases} 1 - \frac{|x_i|}{n} & \text{if } i \in S \\ 1 + \frac{|x_i|}{n} & \text{if } i \notin S \end{cases}
\]
Let $z = y / \|y\|$. So $z^T z = 1$ and $z^T 1 = 0$
Thus \( \lambda_2 \leq z^T (L) z = \frac{y^T L y}{y^T y} = \frac{|\partial S|}{y^T y} \)

Can now show: \( y^T y = \frac{1}{n^2} |S||S'| \).

Result then follows \( \Box \)

Let $G$ be a graph with vertex set $1, 2, \ldots, n$.
A representation for $G$ is a function $F: \{1, 2, \ldots, n\} \to \mathbb{R}^k$ for some $k$.
It would be nice to have adjacent vertices close to each other.
The energy of $F$ is \( \text{Energy of } F = \sum_{i-j} \| F(i) - F(j) \|^2 \)
Minimizing the energy would intuitively guarantee that $F(i)$ is close to $F(j)$ when $i - j$
We can minimize this by taking $F$ to be constant, but this isn't good! We need to place some more constraints on the $F$ we consider.

$$F: \{1, 2, 3, 4, 5, 6\} \rightarrow \mathbb{R}^2$$

$$F(j) = (\cos\left(\frac{2\pi}{6} j\right), \sin\left(\frac{2\pi}{6} j\right))$$

Constraints:
1. $\sum_{i=1}^{n} F(i) = 0$

2. Different additional constraints that have been used.

First is the orthogonality constraint:

We can think of $F$ as an $n \times k$ matrix: $\begin{bmatrix} F(1) \\ \vdots \\ F(n) \end{bmatrix} = M_F$

$F$ is orthogonal if the columns of $M_F$ are orthonormal.

(I.e. $M_F^T M_F = I$)

Q: minimize Energy($F$) over all representations of $G$ that are orthogonal and satisfy 1.

Then (Tutte)

All such $F$ that are optimal have the form

$$M_F = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}$$

where $x_i$ is an e-vector

of $L(G)$ corresponding to $\lambda_i + 1$.

(for example, the picture for the cube graph: \[ \square \])

The second condition that has been studied is to fix a given subset $S$ of vertices.

Now we minimize Energy of $F$ given that $F(i)$ is specified for $i \in S$. Tutte has shown that:

(a) Unique solution to min problem exists

(b) Each unspecified vertex will be the barycenter (center of mass) of its neighbors.
Nonnegative Matrices.

An n×n matrix

$A \geq 0$ if each entry of $A$ is nonnegative.

$A > 0$ if each entry of $A$ is positive.

Q: What can you say about the eigenvalues of $A > 0$?