Homework 1.

**Exercise 1** Describe the solutions as functions of time $t \in \mathbb{R}$ and the phase portraits in the plane $\mathbb{R}^2$ of $\dot{x} = Ax$ for

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

What are the relations between the corresponding solutions?

**Exercise 2** Describe the solutions as functions of time $t \in \mathbb{R}$ and the phase portraits in the plane $\mathbb{R}^2$ of $\dot{x} = Ax$ for

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$ 

**Exercise 3** Compute the solutions of $\dot{x} = Ax$ for

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$ 

**Exercise 4** Determine the stable, center, and unstable subspaces associated with the matrix

$$A = \begin{pmatrix} 3 & -4 & 1 \\ 1 & 0 & -1 \\ -1 & 4 & -3 \end{pmatrix}.$$ 

**Exercise 5** Show that for a matrix $A \in \text{gl}(d, \mathbb{R})$ and $T > 0$ the spectrum $\sigma(e^{AT})$ is given by $\{e^{\lambda T}, \lambda \in \sigma(A)\}$. Show also that the maximal dimension of a Jordan block for $\mu \in \sigma(e^{AT})$ is given by the maximal dimension of a Jordan block of an eigenvalue $\lambda \in \sigma(A)$ with $e^{\lambda T} = \mu$. (Take into account that $e^{i\omega T} = e^{i\omega'} T$ for real $\omega, \omega'$ does not imply $\omega = \omega'$).

As an example, discuss the eigenspace for the eigenvalue 1 of $e^{AT}$ and the eigenspaces of $A$ with

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$
Exercise 6  Consider the following differential equation:

\[ m \ddot{y}(t) + c \dot{y}(t) + ky(t) = 0 \]

where \( m, c, k > 0 \) are constants. Determine all solutions in the following three cases: (i) \( c^2 - 4km > 0 \) (ii) \( c^2 - 4km = 0 \) (iii) \( c^2 - 4km < 0 \). Show that in all cases all solutions tend to the origin as \( t \to \infty \) (the system is asymptotically stable). Determine (in each of the cases (i) to (iii)) the solution \( \varphi(\cdot) \) with \( \varphi(0) = 1 \) and \( \dot{\varphi}(0) = 0 \). Show that in case (iii) all solutions can be written in the form \( \varphi(t) = Ae^{\alpha t} \cos(\beta t - \vartheta) \). Determine \( \alpha \) and \( \beta \).