1. Let $A$ be a $n \times n$ hermitian matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Show that $x^*Ax = x^*x$ if $x$ is an eigenvector of $A$ corresponding to $\lambda_n$, or $x = 0$.

2. Let $G$ be a graph on $n$ vertices and $e$ edges. Let $A$ be the adjacency matrix of $G$. Show that $\lambda_n(A) \geq \frac{2m}{\sqrt{n}}$.

3. Let $A$ be an $n \times n$ hermitian matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and $B$ the $\begin{pmatrix} n-2 \times (n-2) \end{pmatrix}$ leading submatrix of $A$.

   (a) Show $B$ has eigenvalue $\lambda_j$ if and only if $A$ has an eigenvector corresponding to $\lambda_j$ whose first coordinate is 0.

   (b) Show that the multiplicity of $\lambda$ as an eigenvalue of $A$, $m_\lambda(A)$, and the multiplicity of $\lambda$ as an eigenvalue of $B$, $m_\lambda(B)$, satisfy $|m_\lambda(A) - m_\lambda(B)| \leq 1$. 
4. Let $G = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

(a) Find the Laplacian $L$ of $G$

(b) Find the number of spanning trees of $G$ by brute force

(c) Find the determinant of any 3x3 submatrix of $L$, and compare this to b.

5. Let $G$ be a graph on $n$ vertices and $L$ its Laplacian.

Show $x^T L x \geq (x_i - x_j)^2$ for some $i$ adjacent to $j$.

6. Show that $G$ is connected if and only if the second smallest eigenvalue $\lambda_2(L)$ of its Laplacian is 0.

7. Determine the number of spanning trees of the complete graph $K_n$.

8. Compute the number of spanning trees of $K_{m,n}$. 