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Abstract

This course provides an introduction to the interplay between linear algebra and differential equations/dynamical systems in continuous time. We first introduce linear differential equations in Euclidian space $\mathbb{R}^d$ and on Grassmannian and flag manifolds induced by a single matrix $A$, with emphasis on characterizations of the constant matrix $A$ from a dynamics point of view. We then introduce linear skew product flows as a way to model time varying linear systems $\dot{x} = A(t)x$ with, e.g., periodic, measurable ergodic, and continuous chain transitive time dependencies. We develop generalizations of (real parts of) eigenvalues and eigenspaces as a starting point for a linear algebra for classes of time varying linear systems, namely periodic, random, and robust systems. Finally we present some basic ideas to study genuinely nonlinear systems via linearization, emphasizing invariant manifolds and Grobman-Hartman type results that compare nonlinear behavior locally to the behavior of associated linear systems. We will conclude with some engineering applications of this circle of ideas.

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1 Introduction

Linear algebra plays a key role in the theory of dynamical systems, and concepts from dynamical systems allow the study, characterization and generalization of many objects in linear algebra, such as similarity of matrices, eigenvalues, and (generalized) eigenspaces. The most basic form of this interplay can be seen as a matrix $A$ gives rise to a discrete time dynamical system $x_{k+1} = Ax_k$, $k = 0, 1, 2, \ldots$ and to a continuous time dynamical system via the linear ordinary differential equation $\dot{x} = Ax$. In this course we will concentrate on the continuous time case, which actually is in many respects simpler than the discrete time case.

The (real) Jordan form of the matrix $A$ allows us to write the solution of the differential equation $\dot{x} = Ax$ explicitly in terms of the matrix exponential, and hence the properties of the solutions are intimately related to the properties of the matrix $A$. Vice versa, one can consider properties of a linear flow in $\mathbb{R}^d$ and infer characteristics of the underlying matrix $A$. Going one step further, matrices also define (nonlinear) systems on smooth manifolds, such as the sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$, the Grassmann manifolds, the flag manifolds, or on classical (matrix) Lie groups. Again, the behavior of such systems is closely related to matrices and their properties, compare [7].

Since A.M. Lyapunov’s thesis [30] in 1892 it has been an intriguing problem how to construct an appropriate linear algebra for time varying systems. Note that, e.g., for stability of the solutions of $\dot{x} = A(t)x$ it is not sufficient that for all $t \in \mathbb{R}$ the matrices $A(t)$ have only eigenvalues with negative real part (see [23], Chapter 62). Of course, Floquet theory (see [18]) gives an elegant solution for the periodic case, but it is not immediately clear how to build a linear algebra around Lyapunov’s ‘order numbers’ (now called Lyapunov exponents) for more general time dependencies. The multiplicative ergodic theorem of Oseledets [32] resolves some issues for measurable linear systems with stationary time dependencies, and the Morse spectrum together with Selgrade’s theorem [34] goes a long way in describing the situation for continuous linear systems with chain transitive time dependencies.

A third important area of interplay between dynamics and linear algebra arises in the linearization of nonlinear systems about fixed points or arbitrary trajectories. Linearization of a differential equation $\dot{y} = f(y)$ in $\mathbb{R}^d$ about a fixed point $y_0 \in \mathbb{R}^d$ results in the linear differential equation $\dot{x} = D_yf(y_0)x$ and theorems of the type Grobman-Hartman (see, e.g. [10]) resolve the behavior of the flow of the nonlinear equation locally around $y_0$ up to conjugacy, with similar results for dynamical systems over a stochastic or chain-recurrent base.

These observations have important applications in the natural sciences and in engineering design and analysis of systems. Specifically, they are the basis for stochastic bifurcation theory (see, e.g., [3]), and robust stability and stabilizability (see, e.g. [14]). E.g., robust stability radii describe the amount of perturbation the operating point of a system can sustain while remaining stable, and stochastic stability characterizes the limits of acceptable noise in a system, e.g. an electric power system with a substantial component of wind or wave based generation.

This course provides a first introduction to the interplay between linear algebra and differential equations/dynamical systems in continuous time. The first part deals with linear differential equations in Euclidian space $\mathbb{R}^d$ and on Grassmannian and flag manifolds induced by a single matrix $A$, with emphasis on characterizations of the matrix $A$ from a dynamics point of view. The second part introduces linear skew product flows as a way to model time varying linear systems $\dot{x} = A(t)x$ with, e.g., periodic, measurable ergodic, and continuous chain transitive time dependencies. We develop generalizations of (real parts of) eigenvalues and eigenspaces as a starting point for a linear algebra for classes of time varying linear systems, namely periodic, random, and robust systems. The third part introduces some basic ideas to study genuinely nonlinear systems via linearization, emphasizing invariant manifolds and Grobman-Hartman type results that compare nonlinear behavior locally to the behavior of associated linear systems. Several examples of (low-dimensional) systems that play a role in engineering are presented throughout the course.
Notation. Throughout this text we will use the following notation.

\begin{align*}
gl(d, \mathbb{R}) & \quad \text{Lie algebra of real } d \times d \text{ matrices} \\
Gl(d, \mathbb{R}) & \quad \text{Lie group of invertible real } d \times d \text{ matrices} \\
C^k(X,Y) & \quad k\text{-times differentiable functions between } C^k\text{-manifolds } X \text{ and } Y, \\
& \quad \text{with } C^0 \text{ meaning 'continuous'} \\
G_k & \quad k\text{-th Grassmannian manifold of } \mathbb{R}^d \\
A^T & \quad \text{transpose of a matrix } A \in gl(d, \mathbb{R}) \\
\| \cdot \| & \quad \text{any norm on } \mathbb{R}^d
\end{align*}

Part I

Matrices and Linear Dynamical Systems

2 Linear Differential Equations

Linear differential equations can be solved explicitly if one knows the eigenvalues and a basis of eigenvectors (and generalized eigenvectors, if necessary). The key idea is that of the (real) Jordan form of a matrix. The real parts of the eigenvectors determine the exponential behavior of the eigenvectors (and generalized eigenvectors, if necessary). The key idea is that of the (real) Jordan form of a matrix. The real parts of the eigenvectors determine the exponential behavior of the solutions, described by the Lyapunov exponents and the corresponding Lyapunov subspaces. In this section we recall the necessary concepts and results from linear algebra and linear differential equations.

Definition 2.1 For a matrix \( A \in gl(d, \mathbb{R}) \) the exponential \( e^A \in Gl(d, \mathbb{R}) \) is defined by \( e^A = I + \sum_{n=1}^{\infty} \frac{1}{n!} A^n \in Gl(d, \mathbb{R}) \), where \( I \in gl(d, \mathbb{R}) \) is the identity matrix.

A linear differential equation (with constant coefficients) is given by a matrix \( A \in gl(d, \mathbb{R}) \) via \( \dot{x}(t) = Ax(t) \), where \( \dot{x} \) denotes differentiation with respect to \( t \). Any differentiable function \( x: \mathbb{R} \to \mathbb{R}^d \) such that \( \dot{x}(t) = Ax(t) \) for all \( t \in \mathbb{R} \) is called a solution of \( \dot{x} = Ax \).

The initial value problem for a linear differential equation \( \dot{x} = Ax \) consists in finding, for a given initial value \( x_0 \in \mathbb{R}^d \), a solution \( x(\cdot, x_0) \) that satisfies \( x(0, x_0) = x_0 \).

The solutions of a linear, time-invariant differential equation satisfy the following properties, compare e.g. [2] or [26].

Theorem 2.2 For each \( A \in gl(d, \mathbb{R}) \) the solutions of \( \dot{x} = Ax \) form a \( d \)-dimensional vector space \( \text{sol}(A) \subset C^\infty(\mathbb{R}, \mathbb{R}^d) \) over \( \mathbb{R} \), where \( C^\infty(\mathbb{R}, \mathbb{R}^d) = \{ f : \mathbb{R} \to \mathbb{R}^d, f \text{ is infinitely often differentiable} \} \). Note that the solutions of \( \dot{x} = Ax \) are even real analytic.

For each initial value problem given by \( A \in gl(d, \mathbb{R}) \) and \( x_0 \in \mathbb{R}^d \), the solution \( x(\cdot, x_0) \) is unique and given by \( x(t, x_0) = e^{At} x_0 \).

Let \( v_1, \ldots, v_d \in \mathbb{R}^d \) be a basis of \( \mathbb{R}^d \), then the functions \( x(\cdot, v_1), \ldots, x(\cdot, v_d) \) form a basis of the solution space \( \text{sol}(A) \). The matrix function \( X(\cdot) := [x(\cdot, v_1), \ldots, x(\cdot, v_d)] \) is called a fundamental matrix of \( \dot{x} = Ax \), and it satisfies \( X(t) = AX(t) \).

Proof. Using the series expression of \( e^{tA} \) one finds that the matrix \( e^{tA} \) satisfies (in \( \mathbb{R}^d \)) \( \frac{d}{dt} e^{tA} = e^{tA} A \). Hence \( e^{At} x_0 \) is a solution of the initial value problem. To see that the solution is unique, let
\(x(t)\) be any solution of the initial value problem. and put \(y(t) = e^{-tA}x(t)\). Then by the product rule
\[
\dot{y}(t) = \left(\frac{d}{dt} e^{-tA}\right) x(t) + e^{-tA} \dot{x}(t) = -Ae^{-tA}x(t) + e^{-tA}Ax(t) = e^{-tA}(-A + A)x(t) = 0.
\]

Therefore \(y(t)\) is a constant. Setting \(t = 0\) shows \(y(t) = x_0\), and uniqueness follows. The claims on the solution space follow by noting that for every \(t \in \mathbb{R}\) the map
\[
x_0 \mapsto x(t, x_0) : \mathbb{R}^d \to \mathbb{R}^d
\]
is a linear isomorphism. Hence also the map
\[
x_0 \mapsto x(\cdot, x_0) : \mathbb{R}^d \to \text{sol}(A)
\]
is a linear isomorphism. °

The key to obtaining explicit solutions of linear, time-homogeneous differential equations \(\dot{x} = Ax\) are the eigenvalues, eigenvectors, and the real Jordan form of the matrix \(A\).

**Definition 2.3** The distinct (complex) eigenvalues of \(A \in \mathfrak{gl}(d, \mathbb{R})\) will be denoted \(\mu_1, \ldots, \mu_r\). The real versions of the generalized eigenspaces are denoted by \(E(A, \mu_k) \subset \mathbb{R}^d\) or simply \(E_k\) for \(k = 1, \ldots, r \leq d\).

The real Jordan form of a matrix \(A \in \mathfrak{gl}(d, \mathbb{R})\) is denoted by \(J^R_A\). Note that for any matrix \(A\) there is a matrix \(T \in \mathfrak{gl}(d, \mathbb{R})\) such that \(A = T^{-1}J^R_AT\).

With these notations we obtain for the solutions of \(\dot{x} = Ax\) the following result.

**Theorem 2.4** Let \(A \in \mathfrak{gl}(d, \mathbb{R})\) with distinct eigenvalues \(\mu_1, \ldots, \mu_r \in \mathbb{C}\) and corresponding multiplicities \(n_k = \alpha(\mu_k), k = 1, \ldots, r\). If \(E_k\) is a corresponding generalized real eigenspace, then \(\dim E_k = n_k\) and \(\bigoplus_{k=1}^r E_k = \mathbb{R}^d\), i.e. every matrix has a set of generalized real eigenvectors that form a basis of \(\mathbb{R}^d\).

If \(A = T^{-1}J^R_AT\), then \(e^{At} = T^{-1}e^{J^R_AT}T\), i.e. for the computation of exponentials of matrices it is sufficient to know the exponentials of Jordan form matrices.

Let \(v_1, \ldots, v_d\) be a basis of generalized real eigenvectors of \(A\). If \(x_0 = \sum_{i=1}^d \alpha_i v_i\), then \(x(t, x_0) = \sum_{i=1}^d \alpha_i x(t, v_i)\) for all \(t \in \mathbb{R}\). This reduces the computation of solutions to \(\dot{x} = Ax\) to the computation of solutions for Jordan blocks, see the examples below or [26], Chapter 5 for a discussion of this topic.

Each generalized real eigenspace \(E_k\) is invariant for the linear differential equation \(\dot{x} = Ax\), i.e. for \(x_0 \in E_k\) it holds that \(x(t, x_0) \in E_k\) for all \(t \in \mathbb{R}\).

Using this theorem and the real Jordan form of a matrix \(A\) it is possible to give formulas for the solutions \(x(t, x_0) = e^{At}x_0\) of \(\dot{x} = Ax\).

**Example 2.5** Let \(e_1 = (1, 0, \ldots, 0)^T, \ldots, e_d = (0, 0, \ldots, 1)^T\) be the standard basis of \(\mathbb{R}^d\), then \(\{x(\cdot, e_1), \ldots, x(\cdot, e_d)\}\) is a basis of the solution space \(\text{sol}(A)\).

**Example 2.6** Let \(A = \text{diag}(a_1, \ldots, a_d)\) be a diagonal matrix. Then the standard basis \(\{e_1, \ldots, e_d\}\) of \(\mathbb{R}^d\) consists of eigenvectors of \(A\). For \(d = 2\) and \(A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\), one can draw the solutions \(x(t, x_0) \in \mathbb{R}^2\) either componentwise as functions of \(t \in \mathbb{R}\) or as parametrized curves in \(\mathbb{R}^2\). The latter representation of solution is the phase portrait of the differential equation. Observe that by uniqueness of solutions, solution curves cannot intersect.

**Example 2.7** Let \(A = \text{diag}(a_1, \ldots, a_d)\) be a diagonal matrix, then the solution of the linear differential equation \(\dot{x} = Ax\) with initial value \(x_0 \in \mathbb{R}^d\) is given by
\[
e^{At}x_0 = \begin{bmatrix} e^{a_1t} \\ \vdots \\ e^{a_dt} \end{bmatrix} x_0.
\]
Example 2.8 Let $A$ be diagonalizable, i.e. there exists a transformation matrix $T \in GL(d, \mathbb{R})$ and a diagonal matrix $D \in GL(d, \mathbb{R})$ with $A = T^{-1}DT$, then the solution of the linear differential equation $\dot{x} = Ax$ with initial value $x_0 \in \mathbb{R}^d$ is given by $x(t, x_0) = T^{-1}e^{Dt}Tx_0$, where $e^{Dt}$ is given in Example 2.7.

Example 2.9 Let $B = \begin{bmatrix} \lambda & -\nu \\ \nu & \lambda \end{bmatrix}$ be the real Jordan block associated with a complex eigenvalue pair $\mu = \lambda \pm i\nu$ of the matrix $A \in GL(d, \mathbb{R})$. Let $y_0 \in E(A, \mu)$, the real eigenspace of $\mu$. Then the solution $y(t, y_0)$ of $\dot{y} = By$ is given by $y(t, y_0) = e^{\lambda t} \begin{bmatrix} \cos \nu t & -\sin \nu t \\ \sin \nu t & \cos \nu t \end{bmatrix} y_0$. According to Theorem 2.4 this is also the $E(A, \mu)$-component of the solutions of $\dot{x} = J^B_A x$.

Example 2.10 Let $B$ be a Jordan block of dimension $n$ associated with the real eigenvalue $\mu$ of a matrix $A \in GL(d, \mathbb{R})$. Then

$$B = \begin{bmatrix} \mu & 1 & & \\ & \ddots & & \\ & & \ddots & 1 \\ & & & \mu \end{bmatrix}$$

one has $e^{Bt} = e^{\mu t}$

$$e^{\mu t} \begin{bmatrix} 1 \\ t \\ \frac{t^2}{2!} \\ \vdots \\ \frac{t^{n-1}}{(n-1)!} \end{bmatrix}.$$

In other words, for $y_0 = [y_1, \ldots, y_n]^T \in E(A, \mu)$ the $j$-th component of the solution of $\dot{y} = By$ reads

$$y_j(t, y_0) = e^{\mu t} \sum_{k=j}^{n} \frac{t^{k-j}}{(k-j)!} y_k.$$

Example 2.11 Let $B$ be a real Jordan block of dimension $n = 2m$ associated with the complex eigenvalue $\mu = \lambda + i\nu$ of a matrix $A \in GL(d, \mathbb{R})$. Then with $D = \begin{bmatrix} \lambda & -\nu \\ \nu & \lambda \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, for

$$B = \begin{bmatrix} D & I \\ \vdots & \ddots \\ & & I & D \end{bmatrix}$$

one has $e^{Bt} = e^{\lambda t}$

$$e^{\lambda t} \begin{bmatrix} \hat{D} & t\hat{D} & \frac{t^2}{2!} \hat{D} & \cdots & \frac{t^{n-1}}{(n-1)!}\hat{D} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \hat{D} \\ & & & \hat{D} & \hat{D} \end{bmatrix}.$$
Definition 2.12 Let \( x(\cdot, x_0) \) be a solution of the linear differential equation \( \dot{x} = Ax \). Its **Lyapunov exponent** for \( x_0 \neq 0 \) is defined as \( \lambda(x_0) = \limsup_{t \to -\infty} \frac{1}{t} \ln \|x(t, x_0)\| \), where \( \ln \) denotes the natural logarithm and \( \| \cdot \| \) is any norm in \( \mathbb{R}^d \).

Let \( \mu_k = \lambda_k + iv_k, k = 1, \ldots, r \), be the distinct eigenvalues of \( A \in \text{gl}(d, \mathbb{R}) \). We order the distinct real parts of the eigenvalues as \( \lambda_1 < \ldots < \lambda_l \), \( 1 \leq l \leq r \leq d \), and define the **Lyapunov space** of \( \lambda_j \) as \( L(\lambda_j) = \bigoplus E_k \), where the direct sum is taken over all generalized real eigenspaces associated to eigenvalues with real part equal to \( \lambda_j \). Note that \( \bigoplus_{j=1}^l L(\lambda_j) = \mathbb{R}^d \).

The **stable**, **center**, and **unstable subspaces** associated with the matrix \( A \in \text{gl}(d, \mathbb{R}) \) are defined as \( L^- = \bigoplus \{L(\lambda_j), \lambda_j < 0\} \), \( L^0 = \bigoplus \{L(\lambda_j), \lambda_j = 0\} \), and \( L^+ = \bigoplus \{L(\lambda_j), \lambda_j > 0\} \), respectively.

The following result clarifies the relationship between the Lyapunov exponents of \( \dot{x} = Ax \) and the real parts of the eigenvalues of \( A \).

**Theorem 2.13** The Lyapunov exponent \( \lambda(x_0) \) of a solution \( x(\cdot, x_0) \) (with \( x_0 \neq 0 \)) satisfies \( \lambda(x_0) = \lim_{t \to -\infty} \frac{1}{t} \ln \|x(t, x_0)\| = \lambda_j \) if and only if \( x_0 \in L(\lambda_j) \). Hence, associated to a matrix \( A \in \text{gl}(d, \mathbb{R}) \) are exactly 1 Lyapunov exponents, the distinct real parts of the eigenvalues of \( A \).

**Proof.** For any matrix \( A \) there is a matrix \( T \in GL(d, \mathbb{R}) \) such that \( A = T^{-1}J^R \lambda T \), where \( J^R \lambda \) is the real Jordan canonical form of \( A \). Hence the exponential stability behavior of the solutions of \( \dot{x} = Ax \) can be read off from the diagonal elements of \( J^R \lambda \) (cp. Exercise). Hence we may assume that \( A \) is given in real Jordan form. Then the assertions of the theorem can be read off from the solution formulas (2.1) and (2.2) in the generalized eigenspaces.

Using the concept of Lyapunov exponents and Theorem 2.13 we can describe the behavior of solutions of linear differential equations \( \dot{x} = Ax \) as time tends to infinity. By definition, a solution with negative Lyapunov exponent tends to the origin as time tends to infinity, and a solution with positive Lyapunov exponent becomes unbounded as time tends to infinity (the converse need not be true, as we will see in a moment).

It is convenient to formulate the relevant stability concepts not just for linear differential equations, but for general nonlinear differential equations of the form

\[
\dot{x} = f(x),
\]

where \( f : \mathbb{R}^d \to \mathbb{R}^d \) is Lipschitz continuous, i.e. for every \( x \in \mathbb{R}^d \) there are an \( \varepsilon \)-neighborhood \( N(x, \varepsilon) := \{y \in \mathbb{R}^d, \|y - x\| < \varepsilon\} \) with \( \varepsilon > 0 \) and a Lipschitz constant \( L > 0 \) such that

\[
\|f(y) - f(x)\| \leq L \|y - x\| \quad \text{for all } y \in N(x, \varepsilon).
\]

Lipschitz continuity guarantees that there are unique solutions \( \varphi(t, x_0) \) of the initial value problem

\[
\dot{x} = f(x), \quad x(0) = x_0 \in \mathbb{R}^d.
\]

In general, these solutions are only defined on an open interval containing \( t = 0 \). In the following, we will always assume that solutions are defined globally, i.e. for all \( t \in \mathbb{R} \).

Different stability concepts characterize the asymptotic behavior of \( \varphi(t, x_0) \) for \( t \to \pm \infty \).

**Definition 2.14** Let \( x^* \in \mathbb{R}^d \) be a fixed point of the differential equation \( \dot{x} = f(x) \), i.e. \( \varphi(t, x^*) \equiv x^* \) is a solution of \( \dot{x} = f(x) \), where \( \varphi(t, x^*) \) denotes the solution with initial value \( \varphi(0, x^*) \equiv x^* \). Then the point \( x^* \) is called

- **stable** if for all \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that \( \varphi(t, x_0) \in N(x^*, \varepsilon) \) for all \( t \geq 0 \) whenever \( x_0 \in N(x^*, \delta) \),

- **asymptotically stable** if it is stable and there exists a \( \gamma > 0 \) such that \( \lim_{t \to -\infty} \varphi(t, x_0) = x^* \) whenever \( x_0 \in N(x^*, \gamma) \),

- **exponentially stable** if it is asymptotically stable and if there exist \( \alpha, \beta, \) and \( \eta > 0 \) such that for all \( x_0 \in N(x^*, \eta) \) the solution satisfies \( \|\varphi(t, x_0)\| \leq \alpha \|x_0\| e^{-\beta t} \) for all \( t \geq 0 \),

- **unstable** if it is not stable.
It is immediate to see that a point \( x^* \) is a fixed point of \( \dot{x} = Ax \) iff \( x^* \in \ker A \), the kernel of \( A \). The origin \( 0 \in \mathbb{R}^d \) is always a fixed point of any linear differential equation. The following theorem characterizes asymptotic and exponential stability of the origin for \( \dot{x} = Ax \) in terms of the eigenvalues of \( A \).

**Theorem 2.15** For a linear differential equation \( \dot{x} = Ax \) in \( \mathbb{R}^d \) the following statements are equivalent:

(i) The origin \( 0 \in \mathbb{R}^d \) is asymptotically stable.

(ii) The origin \( 0 \in \mathbb{R}^d \) is exponentially stable.

(iii) All Lyapunov exponents (hence all real parts of the eigenvalues) are negative.

(iv) The stable subspace \( L^- \) satisfies \( L^- = \mathbb{R}^d \).

**Proof.** First observe that by linearity, asymptotic and exponential stability of the fixed point \( x^* = 0 \in \mathbb{R}^d \) in a neighborhood \( N(x^*, \gamma) \) implies asymptotic and exponential stability for all points \( x_0 \in \mathbb{R}^d \): Suppose exponential stability holds in \( N(0, \gamma) \). Then for \( x_0 \in \mathbb{R}^d \) the point \( x_1 := \frac{1}{2} x_0 \in N(0, \gamma) \), and hence

\[
\|\varphi(t, x_0)\| = e^{At} x_0 = \frac{2}{\gamma} \|x_0\| e^{At} \frac{x_0}{2} \|x_0\| = \frac{2}{\gamma} \|x_0\| \|\varphi(t, x_1)\| \leq \frac{2}{\gamma} \|x_0\| \alpha \|x_1\| e^{-\beta t} = \alpha \|x_1\| e^{-\beta t},
\]

and analogously for asymptotic stability.

Clearly, properties (ii), (iii) and (iv) are equivalent and imply (i). Conversely, suppose that one of the Lyapunov exponents is nonnegative. Thus one of eigenvalues, say \( \mu \), has nonnegative real part. If \( \mu \) is real, i.e., \( \mu = 0 \), the solution corresponding to the eigenvector in \( \mathbb{R}^d \) is a fixed point and does not tend to the origin as time tends to infinity. Else consider the solution (2.2) in the two-dimensional eigenspace corresponding to the complex eigenvalue pair \( \mu, \bar{\mu} \). This solution also does not tend to the origin as time tends to infinity. Hence (i) implies (iii).

**Remark 2.16** In particular, the proof above shows that for linear systems ‘local stability = global stability’.

It remains to characterize stability of the origin.

**Theorem 2.17** The origin \( 0 \in \mathbb{R}^d \) is stable for the linear differential equation \( \dot{x} = Ax \) iff all Lyapunov exponents are nonpositive, and the eigenvalues with real part zero have a complete set of eigenvectors, i.e. \( L^0 \) is spanned by (real) eigenvectors.

**Proof.** We only have to discuss eigenvalue with zero real part. Suppose first that \( 0 \in \sigma(A) \). Then the solution formula (2.1) shows that an eigenvector yield stable solutions. For a high-dimensional Jordan block, consider \( y_0 = [y_1, ..., y_n]^T = [0, ..., 0, 1]^T \in E(A, \mu) \). Then

\[
y_1(t, y_0) = e^{it} \sum_{k=1}^{n} \frac{t^{k-1}}{(k-1)!} y_k = \frac{t^{n-1}}{(n-1)!} \rightarrow \infty \text{ for } t \rightarrow \infty.
\]

Similarly, one argues for a complex-conjugate pair of eigenvalues.

**Remark 2.18** Eigenvalues \( \mu \) with a complete set of eigenvectors are called semisimple. Equivalently, the corresponding real Jordan blocks are one-dimensional if \( \mu \) is real and two-dimensional, if \( \mu, \bar{\mu} \in \mathbb{C} \setminus \mathbb{R} \). (all complex Jordan blocks are one-dimensional).

### 3 Linear Dynamical Systems in \( \mathbb{R}^d \)

In this section we will look at linear differential equation \( \dot{x} = Ax \), where \( A \in \mathfrak{gl}(d, \mathbb{R}) \), from the point of view of (continuous time) dynamical systems, or linear flows in \( \mathbb{R}^d \). We first introduce (continuous) dynamical systems and the standard concepts for their comparison, namely equivalences and conjugacies that map trajectories into trajectories. For linear flows in \( \mathbb{R}^d \) these concepts lead to two different classifications of matrices, depending on the smoothness of the conjugacy or equivalence. All ideas and results of this section can be found, e.g., in [26] and [33].
Definition 3.1 A continuous dynamical system over the ‘time set’ $\mathbb{R}$ with state space $M$, a complete metric space, is defined as a map $\Phi : \mathbb{R} \times M \to M$ with the properties

(i) $\Phi(0, x) = x$ for all $x \in M$,
(ii) $\Phi(s + t, x) = \Phi(s, \Phi(t, x))$ for all $s, t \in \mathbb{R}$ and all $x \in M$,
(iii) $\Phi$ is continuous (in both variables).

The map $\Phi$ is also called a (continuous) flow.

For each $x \in M$ the time-$t$ map is defined as $\varphi_t = \Phi(t, \cdot) : M \to M$. Using time-$t$ maps, properties (i) and (ii) above can be restated as (i)$'$ $\varphi_0 = id$, the identity map on $M$, (ii)$''$ $\varphi_{s+t} = \varphi_s \circ \varphi_t$ for all $s, t \in \mathbb{R}$.

Note that we have defined a dynamical system over the (two-sided) time set $\mathbb{R}$. This implies invertibility of the time-$t$ maps:

Proposition 3.2 Each time-$t$ map $\varphi_t$ has an inverse $(\varphi_t)^{-1} = \varphi_{-t}$, and $\varphi_t : M \to M$ is a homeomorphism, i.e. a continuous bijective map with continuous inverse.

Denote the set of time-$t$ maps again by $\Phi = \{\varphi_t, t \in \mathbb{R}\}$. A dynamical system is a group in the sense that $(\Phi, \circ)$, with $\circ$ denoting composition of maps, satisfies the group axioms, and $\varphi : (\mathbb{R}, +) \to (\Phi, \circ)$, defined by $\varphi(t) = \varphi_t$ is a group homomorphism.

Systems defined over the one-sided time set $\mathbb{R}^+ := \{t \in \mathbb{R}, t \geq 0\}$ satisfy the corresponding semigroup property and their time-$t$ maps need not be invertible.

Standard examples for continuous dynamical systems are given by solutions of differential equations.

Example 3.3 Let $M$ be a $C^\infty$-differentiable manifold and $X$ a $C^\infty$-vector field on $M$ such that the differential equation $\dot{x} = X(x)$ has unique solutions $x(t, x_0)$ for all $x_0 \in M$ and all $t \in \mathbb{R}$, with $x(0, x_0) = x_0$. Then $\Phi(t, x_0) = x(t, x_0)$ defines a dynamical system $\Phi : \mathbb{R} \times M \to M$.

Example 3.4 Linear differential equations: For $A \in gl(d, \mathbb{R})$ the solutions of $\dot{x} = Ax$ form a continuous dynamical system with time set $\mathbb{R}$ and state space $M = \mathbb{R}^d$. Here $\Phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is defined by $\Phi(t, x_0) = x(t, x_0) = e^{At}x_0$.

Two specific types of orbits will play an important role in these notes, namely fixed point and periodic orbits.

Definition 3.5 A fixed point (or equilibrium) of a dynamical system $\Phi$ is a point $x \in M$ with the property $\Phi(t, x) = x$ for all $t \in \mathbb{R}$.

An orbit $\{\Phi(t, x), t \in \mathbb{R}\}$ of a dynamical system $\Phi$ is called periodic if there exists $T \in \mathbb{R}, T > 0$ such that $\Phi(T + s, x) = \Phi(s, x)$ for all $s \in \mathbb{R}$. The infimum of the positive $T \in \mathbb{R}$ with this property is called the period of the orbit. Note that an orbit of period 0 is a fixed point.

If the system is given by a differential equation as in Example 3.3, the fixed points are easily characterized:

Proposition 3.6 A point $x_0 \in M$ is a fixed point of the dynamical system $\Phi$ associated with a differential equation $\dot{x} = X(x)$ if and only if $X(x_0) = 0$.

For linear differential equations as in Example 3.4 we can say a little more.

Proposition 3.7 Fixed points of linear differential equations: A point $x_0 \in \mathbb{R}^d$ is a fixed point of the dynamical system $\Phi$ associated with the linear differential equation $\dot{x} = Ax$ if and only if $x_0 \in \ker A$, the kernel of $A$.

Periodic orbits of linear differential equations: The orbit $\Phi(t, x_0) := x(t, x_0), t \in \mathbb{R}$ is periodic with some period $T > 0$ if $x_0$ is in the eigenspace of a non-zero complex eigenvalue with zero real part.
**Definition 3.9** Let $\Phi, \Psi : \mathbb{R} \times M \to M$ be two continuous dynamical systems of class $C^k$ ($k \geq 0$), i.e. for $k \geq 1$ the state space $M$ is at least a $C^k$-manifold and $\Phi, \Psi$ are $C^k$-maps. The flows $\Phi$ and $\Psi$ are:

(i) $C^k$-equivalent ($k \geq 1$) if there exists a (local) $C^k$ diffeomorphism $h : M \to M$ such that $h$ takes orbits of $\Phi$ onto orbits of $\Psi$, preserving the orientation (but not necessarily parametrization by time), i.e.,

(a) for each $x \in M$ there is a strictly increasing and continuous parametrization map $\tau_x : \mathbb{R} \to \mathbb{R}$ such that $h(\Phi(t, x)) = \Psi(\tau_x(t), h(x))$ or, equivalently,

(b) for all $x \in M$ and $\delta > 0$ there exists $\varepsilon > 0$ such that for all $t \in (0, \delta)$, $h(\Phi(t, x)) = \Psi(t', h(x))$ for some $t' \in (0, \varepsilon)$.

(ii) $C^k$-conjugate ($k \geq 1$) if there exists a (local) $C^k$ diffeomorphism $h : M \to M$ such that $h(\Phi(t, x)) = \Psi(t, h(x))$ for all $x \in M$ and $t \in \mathbb{R}$.

(iii) The flows $\Phi$ and $\Psi$ are $C^0$-equivalent if there exists a (local) homeomorphism $h : M \to M$ satisfying the properties of (i) above, and they are $C^0$-conjugate if there exist a (local) homeomorphism $h : M \to M$ satisfying the properties of (ii) above. Often, $C^0$-equivalence is called topological equivalence, and $C^0$-conjugacy is called topological conjugacy or simply conjugacy.

The first important idea in theory of dynamical systems concerns comparison of two systems, i.e. how can we tell that two systems are 'the same'? This idea is formalized through conjugacies and equivalences, which we define next.

**Remark 3.8** The converse of the second assertion is not true: Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

with eigenvalues $\{1, \pm i\}$. The initial value $x_0 = (1, 1, 0)^T$ leads to the periodic solution with period $t = 1$

$$e^{At} = \begin{pmatrix} 1 \\ \cos t \\ \sin t \end{pmatrix}.$$

Proof. The first assertion is obvious. The second assertion follows from the solution formula (2.2).

**Theorem 3.10** For two linear flows $\Phi$ (associated with $\dot{x} = Ax$) and $\Psi$ (associated with $\dot{x} = Bx$) in $\mathbb{R}^d$, the following are equivalent:

(i) $\Phi$ and $\Psi$ are $C^k$-conjugate for $k \geq 1$,

(ii) $\Phi$ and $\Psi$ are linearly conjugate, i.e., the conjugacy map $h$ is a linear operator in $\text{Gl}(\mathbb{R}^d)$,

(iii) $A$ and $B$ are similar, i.e., $A = TBT^{-1}$ for some $T \in \text{Gl}(d, \mathbb{R})$.

Each of these statements implies that $A$ and $B$ have the same eigenvalue structure and (up to a linear transformation) the same generalized real eigenspace structure. In particular, the $C^k$-conjugacy classes are exactly the real Jordan canonical form equivalence classes in $\text{gl}(d, \mathbb{R})$. 

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Proof. Properties (ii) and (iii) are obviously equivalent and imply (i). Suppose that (i) holds, and let $h : \mathbb{R}^d \to \mathbb{R}^d$ be a $C^k$-conjugacy. Thus for all $x \in \mathbb{R}^d$ and $t > 0$

$$h(\Phi(t, x) = h(e^{At} x) = e^{Bt} h(x) = \Psi(h(x)).$$

Differentiating with respect to $t$ and using the chain rule we find

$$Dh(e^{At} x) A e^{At} = B e^{Bt} Dh(x).$$

Evaluating this at $t = 0$ we get for all $x \in \mathbb{R}^d$

$$Dh(x) A = B Dh(x).$$

Since $h$ is a diffeomorphism, the linear map $Dh(x)$ is invertible. It follows that $H := Dh(x) \in GL(d, \mathbb{R})$ is a linear conjugacy, since $HA = BH$ implies

$$H e^{At} = e^{Bt} H \text{ for all } t \in \mathbb{R}.$$  

\[\Box\]

Similar, but somewhat more involved arguments show the following characterization of $C^k$-equivalence.

Theorem 3.11 For two linear flows $\Phi$ (associated with $\dot{x} = Ax$) and $\Psi$ (associated with $\dot{x} = Bx$) in $\mathbb{R}^d$, the following are equivalent:

(i) $\Phi$ and $\Psi$ are $C^k$-equivalent for $k \geq 1$

(ii) $\Phi$ and $\Psi$ are linearly equivalent, i.e., the equivalence map $h$ is a linear map in $GL(\mathbb{R}^d)$,

(iii) $A = \alpha T B T^{-1}$ for some positive real number $\alpha$ and $T \in GL(d, \mathbb{R})$.

Each of these statements implies that $A$ and $B$ have the same real Jordan structure and their eigenvalues differ by a positive constant. Hence the $C^k$-equivalence classes are real Jordan canonical form equivalence classes modulo a positive constant.

In particular we obtain for the linear dynamical systems induced by a matrix $A$ and its real Jordan form $J_A^R$.

Corollary 3.12 For each matrix $A \in gl(d, \mathbb{R})$ its associated linear flow in $\mathbb{R}^d$ is $C^k$-conjugate (and hence $C^k$-equivalent) for all $k \geq 0$ to the dynamical system associated with the Jordan form $J_A^R$.

Theorem 3.10 clarifies the structure of two matrices that give rise to conjugate or equivalent flows under (global) $C^k$-diffeomorphisms with $k \geq 1$. For homeomorphisms, i.e., for $k = 0$, the situation is quite different and somewhat surprising. To explain the corresponding result we first need to introduce the concept of hyperbolicity.

Definition 3.13 The matrix $A \in gl(d, \mathbb{R})$ is hyperbolic if it has no eigenvalues on the imaginary axis.

The set of hyperbolic matrices in $gl(d, \mathbb{R})$ is rather 'large':

Proposition 3.14 The set of hyperbolic matrices is open and dense in $gl(d, \mathbb{R})$. A matrix $A$ is hyperbolic if and only if it is structurally stable in $gl(d, \mathbb{R})$, i.e., there exists a neighborhood $U \subset gl(d, \mathbb{R})$ of $A$ such that all $B \in U$ are topologically equivalent to $A$.

Proof. Let $A$ be hyperbolic. Then also for all matrices in a neighborhood of $A$ all eigenvalues have nonvanishing imaginary parts, since the eigenvalues depend continuously on the matrix. Hence the assertion follows. $\Box$

With these preparations we can formulate the characterization of $C^0$-conjugacies of linear flows:

Theorem 3.15 If $A$ and $B$ are hyperbolic, then the associated linear flows $\Phi$ and $\Psi$ in $\mathbb{R}^d$ are $C^0$-equivalent (and $C^0$-conjugate) if and only if the dimensions of the stable subspaces (and hence the dimensions of the unstable subspaces) of $A$ and $B$ agree.
We will only prove the assertion for conjugacies. This needs some preparations. Consider first asymptotically stable differential equations \( \dot{x} = Ax \). Thus for all \( \kappa < \max \{ \Re \lambda, \lambda \in \sigma(A) \} \) there is \( C > 0 \) with

\[
|e^{At}x| \leq Ce^{\kappa t} |x| \text{ for all } x \in \mathbb{R}^n, \ t \geq 0;
\]

here we use the Euclidean norm,

\[
|x| = \sqrt{x_1^2 + \ldots + x_n^2}.
\]

This norm need not decrease monotonically along solutions, as shown by the following simple example.

**Example 3.16** Consider in \( \mathbb{R}^2 \)

\[
\dot{x} = -x - y, \\
\dot{y} = 4x - y.
\]

The eigenvalues of \( A = \begin{pmatrix} -1 & -1 \\ 4 & -1 \end{pmatrix} \) are the zeros of \( \det \begin{pmatrix} \lambda + 1 & 1 \\ -4 & \lambda + 1 \end{pmatrix} = (\lambda + 1)^2 + 4 \), hence equal to \(-1 \pm 2i\). The solutions are

\[
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} \cos 2t & -\frac{1}{2} \sin 2t \\ 2 \sin 2t & \cos 2t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\]

The origin is asymptotically stable, but the distance to the origin does not decrease monotonically.

The following proposition shows that monotonicity always holds in a norm adapted to the matrix \( A \).

**Proposition 3.17** For \( \dot{x} = Ax \) with \( A \in \mathfrak{gl}(d, \mathbb{R}) \) the following properties are equivalent:

(i) There are a norm \( \| \cdot \|_* \) on \( \mathbb{R}^d \) and \( a > 0 \) such that for all \( x \in \mathbb{R}^d \)

\[
\|e^{At}x\|_* \leq e^{-at} \|x\|_* \text{ for } t \geq 0.
\]

(ii) For every norm \( \| \cdot \| \) on \( \mathbb{R}^d \) there are \( a > 0 \) and \( c > 0 \) with

\[
\|e^{At}x\| \leq C e^{-at} \|x\| \text{ for } t \geq 0.
\]

(iii) For every eigenvalue \( \lambda \) of \( A \) one has \( \Re \lambda < 0 \).

A norm as in (i) is called adapted norm.

**Proof.** (i) implies (ii), since all norms on \( \mathbb{R}^d \) are equivalent. Property (ii) (with \(-a > \max \Re \lambda\)) is equivalent to (iii) by Theorem 2.15. It remains to show that (ii) implies (i). Let \( b \in (0, a) \). Then (ii) implies for \( t \geq 0 \)

\[
|e^{At}x| \leq c e^{-at} |x| = c e^{(b-a)t} e^{-bt}.
\]

Hence there is \( \tau > 0 \), such that \( c e^{(b-a)t} < 1 \) for all \( t \geq \tau \), hence

\[
|e^{At}x| \leq e^{-bt}.
\]

Then

\[
\|x\|_* := \int_0^\tau e^{bs} |e^{As}x| \ ds, \ x \in \mathbb{R}^d,
\]

defines a norm, since

\[
\|x\|_* = 0 \iff e^{bs} |e^{As}x| \text{ for } s \in [0, \tau] \iff x = 0,
\]

\[
\|x + y\|_* = \int_0^\tau e^{bs} |e^{As}(x + y)| \ ds \leq \|x\|_* + \|y\|_*.
\]
This norm has the desired monotonicity property: For \( t \geq 0 \) write \( t = n \tau + T \) with \( 0 \leq T < \tau \) and

\[
\left\| e^{Ax} \right\|_* = \int_0^T e^{bs} \left| e^{As} e^{At} x \right| \, ds
\]

\[
= \int_0^{\tau - T} e^{bs} \left| e^{A(n+1)s} e^{At} x \right| \, ds + \int_{\tau - T}^T e^{bs} \left| e^{A(n+1)\tau} e^{At} x \right| \, ds
\]

\[
\leq \int_T^\infty e^{b(\sigma - T)} \left| e^{A\sigma x} \right| \, d\sigma + \int_0^T e^{b(\sigma - T + \tau)} \left| e^{A(n+1)\tau} e^{At} x \right| \, ds
\]

with \( \sigma := T + s \) and \( \sigma = T - \tau + s \); respectively. Now (3.1) yields

\[
\leq e^{-bt} \int_0^T e^{b\sigma} \left| e^{A\sigma x} \right| \, d\sigma
\]

\[
eq e^{-bt} \left\| x \right\|_*
\]

and the assertion follows. \( \blacksquare \)

**Remark 3.18** In (i) and (ii) one can choose any \( a < - \max \text{Re} \lambda \). The proposition also yields another proof that exponential stability follows if all eigenvalues have negative real parts.

Next we show the assertion of Theorem 3.15 in the asymptotically stable case.

**Proposition 3.19** Let \( A, B \in \mathfrak{gl}(d, \mathbb{R}) \). If all eigenvalues of \( A \) and of \( B \) have negative real parts, then the flows \( e^{At} \) and \( e^{Bt} \) are topologically conjugate.

**Proof.** Let \( \left\| \cdot \right\|_A \) and \( \left\| \cdot \right\|_B \) be corresponding adapted norms, hence, with \( a, b > 0 \),

\[
\left| e^{At} x \right|_A \leq e^{-at} \left| x \right|_A \quad \text{and} \quad \left| e^{Bt} x \right|_B \leq e^{-bt} \left| x \right|_B \text{ for } t \geq 0 \text{ and } x \in \mathbb{R}^d.
\]

Then for \( t \leq 0 \)

\[
\left| e^{At} x \right|_A \geq e^{a|t|} \left| x \right|_A \quad \text{and} \quad \left| e^{Bt} x \right|_B \geq e^{b|t|} \left| x \right|_B
\]

by applying the inequality above to \( e^{At} \) and \( -t \). Thus

\[
\left| x \right|_A = \left| e^{-At} e^{At} x \right|_A \leq e^{at} \left| e^{At} x \right|_A
\]

and, similarly, for \( B \). We denote the corresponding unit spheres

\[
S_A = \{ x \in \mathbb{R}^n, \left| x \right|_A = 1 \} \quad \text{and} \quad S_B = \{ x \in \mathbb{R}^n, \left| x \right|_B = 1 \}
\]

as fundamental domains of the flows \( e^{At} \) and \( e^{Bt} \). Define a homeomorphisms \( h_0 : S_A \to S_B \) by

\[
h_0(x) = \frac{x}{\left| x \right|_B}
\]

with inverse \( h_0^{-1} : S_B \to S_A \)

\[
h_0^{-1}(y) = \frac{y}{\left| y \right|_A}.
\]

In order to extend this map to \( \mathbb{R}^d \) observe that (by the mean value theorem and by definition of the adapted norms) there is for all \( x \neq 0 \) a unique time \( \tau(x) \in \mathbb{R} \) with \( \left| e^{A\tau(x)} x \right|_A = 1 \). This immediately implies \( \tau(e^{At} x) = \tau(x) - t \). Furthermore, the map \( x \mapsto \tau(x) \) is continuous. Now define \( h : \mathbb{R}^d \to \mathbb{R}^d \) by

\[
h(x) = \begin{cases} e^{-B\tau(x)} h_0(e^{A\tau(x)} x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}
\]

\[
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\]
Then $h$ is a conjugation, since
\[
h(e^{At}) = e^{-Br(e^{At}x)} h_0(e^{Ar(e^{At}x)} e^{At}x) = e^{-B[\tau(x)-t]} h_0(e^{A[\tau(x)-t]} e^{At}x) \\
= e^{Bt} e^{-Br(x)} h_0(e^{Ar(x)} x) \\
= e^{Bt} h(x).
\]

The map $h$ is continuous in $x \neq 0$, since $e^{At}$ and $e^{Bt}$ as well as $\tau(x)$ are continuous. In order to see continuity in $x = 0$, consider a sequence $x_j \to 0$. Then $\tau_j := \tau(x_j) \to -\infty$. Let $y_j := h_0(e^{\tau_j} x)$. Then $|y_j|_B = 1$ and hence
\[
|h(x_j)|_B = |e^{-B\tau_j} y_j|_B \leq e^{B\tau_j} \to 0 \text{ for } j \to \infty.
\]

The map is injective: Suppose $h(x) = h(z)$. The case $x = 0$ is clear. Hence suppose that $x \neq 0$. Then
\[
h(x) = h(z) \neq 0
\]
and with $\tau := \tau(x)$ the conjugation property implies
\[
h(e^{A\tau} x) = e^{B\tau} h(x) = e^{B\tau} h(z) = h(e^{A\tau} z).
\]
Thus $h(e^{A\tau} z) = h(e^{A\tau} x) \in S_B$. Since $h$ maps only $S_A$ to $S_B$, it follows that
\[
e^{A\tau} z \in S_A \text{ and hence } \tau = \tau(x) = \tau(z).
\]
By
\[
h_0(e^{A\tau} x) = h(e^{A\tau} x) = h(e^{A\tau} z) = h_0(e^{A\tau} z)
\]
and injectivity of $h_0$ we find
\[
e^{A\tau} x = e^{B\tau} z, \text{ and hence } x = z.
\]

Exchanging the roles of $A$ and $B$ we see that $h^{-1}$ exists and is continuous. ■

**Proof of Theorem 3.15.** There are conjugations
\[
h^s : E^s_A \to E^s_B \text{ and } h^u : E^u_A \to E^u_B
\]
between the restrictions to the stable and the unstable subspaces of $e^{At}$ and $e^{Bt}$. With the projections
\[
\pi^s : \mathbb{R}^n \to E^s_A \text{ and } \pi^u : \mathbb{R}^n \to E^u_A
\]
a topological conjugation is defined by
\[
h(x) := h^s(\pi^s(x)) + h^u(\pi^u(x)).
\]
■

**Remark 3.20** If $A \in \mathbb{R}^{n \times n}$ is hyperbolic and $B$ is close enough to $A$, then the corresponding flows $e^{At}$ and $e^{Bt}$ are topologically conjugate. This follows, since not only the eigenvalues, but also the generalized eigenspaces depend continuously on the matrix.

**Remark 3.21** Theorems 3.11 and 3.15 characterize matrices via invariance properties of the associated linear autonomous differential equations under continuous or smooth conjugacies. This may be viewed as part of Klein’s Erlanger Programm in the nineteenth century defining geometries by groups of transformations. This point of view is extended by McSwiggen and Meyer in [31] where they also discuss invariance properties under Lipschitz and Hölder conjugacies; see also Kawan and Stender [27] for a classification under Lipschitz conjugacies.
The dynamical systems considered in this section are linear flows \( \Phi : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d \), i.e. they satisfy the condition \( \Phi(t, \alpha x + \beta y) = \alpha \Phi(t, x) + \beta \Phi(t, y) \) for all \( x, y \in \mathbb{R}^d \) and all \( t \in \mathbb{R} \). For flows of this type we obtained the very satisfying characterizations via their matrix generators in Theorems 3.10 and 3.15. However, some important features of matrices and their associated linear differential equations cannot be described by \( C^k \)-conjugacies or equivalences. This includes, e.g., the exponential growth behavior as measured by the real part of the eigenvalues of the matrix generator \( A \), compare Definition 2.12 and Theorem 2.13. These finer properties of matrices require the study of induced dynamical systems on nonlinear spaces. The next section introduces some concepts and results necessary for the analysis of nonlinear dynamical systems. In Section 5 we will apply our insights to obtain additional information on the connections between matrices and dynamical systems.

4 Chain Recurrence and Morse Decompositions of Dynamical Systems

A matrix \( A \in \text{gl}(d, \mathbb{R}) \) and hence a linear differential equation \( \dot{x} = Ax \) maps subspaces of \( \mathbb{R}^d \) into subspaces of \( \mathbb{R}^d \) preserving the dimensions. Therefore the matrix \( A \) also defines dynamical systems on spaces of subspaces, such as the Grassmann and the flag manifolds. These are nonlinear systems, but they can be studied via linear algebra, and vice versa, the behavior of these systems allows for the investigation of certain properties of the matrix \( A \). The key topological concepts for the analysis of systems on compact spaces, like the Grassmann and flag manifolds are chain recurrence, Morse decompositions and attractor-repeller decompositions. This subsection concentrates on the first two approaches, the connection to attractor-repeller decompositions can be found, e.g., in Ayala-Hoffmann et al. [6]. For additional details on the concepts and results of this section we refer the reader to Alongi and Nelson [1], Robinson [33] and Ayala, Colonius, Kliemann [7].

Global analysis of dynamical systems starts with limit sets, i.e. with the question, where do the trajectories go for \( t \rightarrow \pm \infty \). The following definition formalizes this question.

**Definition 4.1** Given a dynamical system \( \Phi : \mathbb{R} \times M \rightarrow M \). For a subset \( N \subset M \) the \( \alpha \)-limit set is defined as \( \alpha(N) = \{ y \in M, \text{there exist sequences} \ x_n \text{in} \ N \text{and} \ t_n \rightarrow -\infty \text{in} \mathbb{R} \text{with} \lim_{n \rightarrow -\infty} \Phi(t_n, x_n) = y \} \), and similarly the \( \omega \)-limit set of \( N \) is defined as \( \omega(N) = \{ y \in M, \text{there exist sequences} \ x_n \text{in} \ N \text{and} \ t_n \rightarrow \infty \text{in} \mathbb{R} \text{with} \lim_{n \rightarrow \infty} \Phi(t_n, x_n) = y \} \).

Note that if \( M \) is compact, then \( \alpha \)-limit sets and \( \omega \)-limit sets exist for all \( N \subset M \). This need not be true in the non-compact case, see the examples below. We look at some examples of limit sets in low-dimensional systems.

**Example 4.2** Dynamical systems in \( \mathbb{R}^1 \): Any limit set \( \alpha(x) \) and \( \omega(x) \) from a single point \( x \) of a dynamical system in \( \mathbb{R}^1 \) consists of a single fixed point. The chain recurrent components (and the finest Morse decomposition) consist of single fixed points or intervals of fixed points. Any Morse set consists of fixed points and intervals between them.

**Example 4.3** Dynamical systems in \( \mathbb{R}^2 \): A non-empty, compact limit set of a dynamical system in \( \mathbb{R}^2 \), which contains no fixed points, is a closed, i.e. a periodic orbit (Poincaré–Bendixson theory; see [26]). Any non-empty, compact limit set of a dynamical system in \( \mathbb{R}^2 \) consists of fixed points, connecting orbits (such as homoclinic or heteroclinic orbits), and periodic orbits.

**Example 4.4** Consider the following dynamical system \( \Phi \) in \( \mathbb{R}^2 \setminus \{0\} \), given by a differential equation in polar form for \( r > 0, \theta \in [0, 2\pi) \), and \( a \neq 0 \):

\[
\dot{r} = 1 - r, \quad \dot{\theta} = a.
\]

For each \( x \in \mathbb{R}^2 \setminus \{0\} \) the \( \omega \)-limit set is the circle \( \omega(x) = S^1 = \{(r, \theta), \ r = 1, \ \theta \in [0, 2\pi)\} \). The state space \( \mathbb{R}^2 \setminus \{0\} \) is not compact, and \( \alpha \)-limit sets only exist for the points \( y \in S^1 \); for these points \( \alpha(y) = S^1 \).

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In many cases, the limit sets, as limits of trajectories, are not too well-behaved, i.e. they may not be isolated. One therefore looks at the concept of chains, which generalizes the idea of trajectories:

**Definition 4.5** For a flow $\Phi$ on a complete metric space $M$ and $\varepsilon, T > 0$ an $(\varepsilon,T)$-**chain** from $x \in M$ to $y \in M$ is given by

$$n \in \mathbb{N}, \quad x_0 = x, \ldots, x_n = y, \quad T_0, \ldots, T_{n-1} > T$$

with

$$d(\Phi(T_i, x_i), x_{i+1}) < \varepsilon \text{ for all } i,$$

where $d$ is the metric on $M$.

A set $K \subset M$ is **chain transitive** if for all $x, y \in K$ and all $\varepsilon, T > 0$ there is an $(\varepsilon,T)$-chain from $x$ to $y$.

The **chain recurrent set** $\text{CR}$ is the set of all points that are chain reachable from themselves, i.e. $\text{CR} = \{ x \in M, \text{ for all } \varepsilon, T > 0 \text{ there is an } (\varepsilon,T)\text{-chain from } x \text{ to } y \}$. A set $\mathcal{M} \subset M$ is a **chain recurrent component**, if it is a maximal (with respect to set inclusion) chain transitive set. In this case $\mathcal{M}$ is a connected component of the chain recurrent set $\text{CR}$.

An example of a flow for which the limits sets from points are strictly contained in the chain recurrent components can be obtained as follows:

**Example 4.6** Let $M = [0,1] \times [0,1]$. Let the flow $\Phi$ on $M$ be defined such that all points on the boundary are fixed points, and the orbits for points $(x,y) \in (0,1) \times (0,1)$ are straight lines $\Phi(\cdot,(x,y)) = \{(z_1, z_2), \quad z_1 = x, \quad z_2 \in (0,1)\}$ with $\lim_{t \to \pm \infty} \Phi(t, (x,y)) = (x, \pm 1)$. For this system, each point on the boundary is its own $\alpha$- and $\omega$-limit set. The $\alpha$-limit sets for points in the interior $(x,y) \in (0,1) \times (0,1)$ are of the form $\{(x,-1)\}$, and the $\omega$-limit sets are $\{(x,+1)\}$. The only chain recurrent component for this system is $M = [0,1] \times [0,1]$ (which is also the only Morse set, compare Definition 4.9).

The following lemma shows that for chain transitivity it is not important to use chains with arbitrary large jump times $T_i$.

**Proposition 4.7** A set $K \subset M$ is chain transitive iff for all $x, y \in K$ and all $\varepsilon > 0$ there is an $(\varepsilon,T)$-chain from $x$ to $y$ with jump times $T_i \in (1/2, 2]$.

**Proof.** If $K \subset M$ is chain transitive, we may trivially introduce artificial jumps. For the converse, it suffices to show the following: Let $x, y \in K$ and let $\tau > 0$. If for every $\varepsilon > 0$ there exists an $(\varepsilon, \tau)$-chain from $x$ to $y$, then for every $\varepsilon, T > 0$ there exists an $(\varepsilon,T)$-chain from $x$ to $y$. This, in turn, follows, if we can show that for every $\varepsilon > 0$ there is an $(\varepsilon,2\tau)$-chain from $x$ to $y$. By compactness of $X$ the map $\Phi$ is uniformly continuous on $X \times [0,3\tau]$. Hence there is $\delta \in (0, \frac{\varepsilon}{2})$ such that for all $a, b \in X$ and $t \in [0,3\tau]$:

$$d(a, b) < \delta \text{ implies } d(\Phi(t,a), \Phi(t,b)) < \frac{\varepsilon}{2}.$$ 

Now let a $(\delta, \tau)$-chain $x_0 = x, x_1, \ldots, x_m = y$ with times $\tau_0, \ldots, \tau_{m-1} \geq \tau$ be given. We may assume that $\tau_i \in [\tau, 2\tau]$. We also may assume that $m \geq 2$, because we may concatenate this chain with a chain from $y$ to $x$. Thus there are $q \in \{0,1,\ldots\}$ and $r \in \{2,3\}$ with $m = 2q + r$. We obtain an $(\varepsilon,2\tau)$-chain from $x$ to $y$ given by points

$$y_0 = x, \quad y_1 = x_2, \quad y_2 = x_4, \ldots, \quad y_q = x_{2q}, \quad y_{q+1} = x_m = y$$

with times

$$T_0 = \sum_{i=0}^{1} \tau_i, \quad T_1 = \sum_{i=2}^{3} \tau_i, \ldots, \quad T_q = \sum_{i=2q}^{m} \tau_i.$$ 

This follows by the triangle inequality and the choice of $\delta$. $\blacksquare$

Using the idea of Morse decompositions one can show that the chain recurrent components cover the global behavior of a dynamical system for compact $M$. For this we first need the definition of isolated invariant sets.
Definition 4.8 For a flow $\Phi$ on a complete metric space $M$, a compact subset $K \subset M$ is called isolated invariant, if it is invariant and there exists a neighborhood $N$ of $K$, i.e., a set $N$ with $K \subset \text{int } N$, such that $\Phi(t, x) \in N$ for all $t \in \mathbb{R}$ implies $x \in K$.

Definition 4.9 A Morse decomposition of a flow $\Phi$ on a complete metric space $M$ is a finite collection $\{M_i, i = 1, \ldots, l\}$ of nonvoid, pairwise disjoint, and isolated compact invariant sets such that

(i) for all $x \in M$, $\omega(x), \alpha(x) \subset \bigcup_{i=1}^{l} M_i$; and

(ii) suppose there are $M_{j_0}, M_{j_1}, \ldots, M_{j_n}$ and $x_1, \ldots, x_n \in M \setminus \bigcup_{i=1}^{l} M_i$ with $\alpha(x_i) \subset M_{j_{i-1}}$ and $\omega(x_i) \subset M_{j_i}$ for $i = 1, \ldots, n$; then $M_{j_0} \neq M_{j_n}$.

The elements of a Morse decomposition are called Morse sets. A Morse decomposition $\{M_1, \ldots, M_n\}$ is called finer than a Morse decomposition $\{M'_1, \ldots, M'_{n'}\}$, if for all $j \in \{1, \ldots, n'\}$ there is $i \in \{1, \ldots, n\}$ with $M_i \subset M'_j$.

For one-dimensional systems the recurrent components and the Morse decompositions are easy to describe.

Example 4.10 Dynamical systems in $\mathbb{R}^1$: The chain recurrent components (and the finest Morse decomposition) consist of single fixed points or intervals of fixed points. Any Morse set consists of fixed points and intervals between them.

The next result shows that Morse decompositions characterize the global behavior of a dynamical system.

Proposition 4.11 For a Morse decomposition $\{M_i, i = 1, \ldots, l\}$ the relation $M_i \prec M_j$, given by $\alpha(x) \subset M_i$ and $\omega(x) \subset M_j$ for some $x \in M \setminus \bigcup_{i=1}^{l} M_i$, induces an order.

This result says that the flow of a dynamical system goes from a lesser (with respect to the order $\prec$) Morse set to a greater Morse set for trajectories that do not start in one of the Morse sets. It remains to clarify the behavior of the system on the Morse sets.

Theorem 4.12 Let $M$ be compact. Then there exists a finest Morse decomposition $\{M_1, \ldots, M_n\}$ of the flow $\Phi$ if and only if the chain recurrent set $\mathcal{R}$ has finitely many connected components. In this case, the Morse sets coincide with the chain recurrent components of $\mathcal{R}$ and the flow restricted to any Morse set $M$ is chain transitive and chain recurrent, i.e., all points $x \in M$ are chain recurrent points.

We have now seen many concepts to describe the qualitative behavior of a dynamical system, including fixed points, periodic (or closed) orbits, limit sets, chain recurrence and Morse decompositions. If these concepts describe intrinsic properties of a system that can be used for its characterization, they should survive under conjugacies and equivalences. The next results show that this is actually true.

Theorem 4.13 Let $\Phi, \Psi : \mathbb{R} \times M \rightarrow M$ be two dynamical systems on a compact metric state space $M$ and let $h : M \rightarrow M$ be a topological equivalence for $\Phi$ and $\Psi$. Then

(i) the point $p \in M$ is a fixed point of $\Phi$ if and only if $h(p)$ is a fixed point of $\Psi$;

(ii) the orbit $\Phi(\cdot, p)$ is closed if and only if $\Psi(\cdot, h(p))$ is closed;

(iii) if $K \subset M$ is an $\alpha$-(or $\omega$-) limit set of $\Phi$ from $p \in M$, then $h[K]$ is an $\alpha$-(or $\omega$-) limit set of $\Psi$ from $h(p) \in M$.

(iv) Given, in addition, two dynamical systems $\Theta_1, \Theta_2 : \mathbb{R} \times N \rightarrow N$. If $h : M \rightarrow M$ is a topological conjugacy for the flows $\Phi$ and $\Psi$ on $M$, and $g : N \rightarrow N$ is a topological conjugacy for $\Theta_1$ and $\Theta_2$ on $N$, then the product flows $\Phi \times \Theta_1$ and $\Psi \times \Theta_2$ on $M \times N$ are topologically conjugate via $h \times g : M \times N \rightarrow M \times N$. This result is, in general, not true for topological equivalence.
The following is an example of two (nonlinear) flows that are topologically equivalent, but not conjugate.

**Example 4.14** Consider again the dynamical system \( \Phi \) in \( \mathbb{R}^2 \setminus \{0\} \), given by a differential equation in polar form for \( r > 0, \theta \in [0, 2\pi) \), and \( a \neq 0 \):

\[
\dot{r} = 1 - r, \quad \dot{\theta} = a,
\]

compare Example 4.4. Define a second system \( \Psi \) given by

\[
\dot{r} = 1 - r, \quad \dot{\theta} = b
\]

with \( b \neq 0 \). Then the flows \( \Phi \) and \( \Psi \) are topologically equivalent, but not conjugate if \( b \neq a \).

Finally we notice the preservation of the concepts defined in this section under equivalences between dynamical systems.

**Theorem 4.15** (i) Topological equivalences (and conjugacies) on a compact metric space \( M \) map chain transitive sets onto chain transitive sets.

(ii) Topological equivalences map invariant sets onto invariant sets, and minimal closed invariant sets onto minimal closed invariant sets.

(iii) Topological equivalences map Morse decompositions onto Morse decompositions; finest Morse decompositions are mapped onto finest Morse decompositions.

**Proof.** (i) The equivalence map \( h \) is a homeomorphism and \( M \) is compact by assumption. Hence for all \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for all \( z \in M \) it holds that \( B(z, \varepsilon) \subset h^{-1}[B(h(z), \delta)] \) with \( B(z, \varepsilon) = \{ y \in M : d(z, y) < \varepsilon \} \). Let \( \varphi_1, \varphi_2 \) be flows on \( M \) with topological conjugacy \( h \). For a chain transitive set \( N_2 \subset M \) of \( \varphi_2 \), we claim that \( N_1 := h^{-1}[N_2] \) is a chain transitive set of \( \varphi_1 \). Take \( p_1, q_1 \in N_1 \) and fix \( \varepsilon > 0, T > 0 \). Choose \( \delta \) as above and let \( \xi_2 \) be a \((\delta, T)\)-chain from \( p_2 = h(p_1) \) to \( q_2 = h(q_1) \). Then \( h^{-1}(\xi_2) =: \xi_1 \) is an \((\varepsilon, T)\)-chain from \( p_1 \) to \( q_1 \). For a topological equivalence \( h \), we need to adjust the time of \( \xi_2 \). Since the time parametrization \( \tau(\cdot) \) of \( \varphi_2 \) with respect to \( \varphi_1 \) is continuous in both variables, we can define \( T_1 = \min_{p \in M} \tau_p(T) \). If we choose \( \xi_2 \) in the proof above to be a \((\delta, T_1)\)-chain, then \( h^{-1}(\xi_2) \) is an \((\varepsilon, T)\)-chain of \( \varphi_1 \) from \( p_1 \) to \( q_1 \).

(ii) This follows, since \( h \) maps orbits onto orbits and closures of orbits onto closures of orbits.

(iii) This follows from Theorem 4.13 (iii) and the assertions above. ■

## 5 Linear Systems on Grassmannian and Flag Manifolds

In this section we return to matrices \( A \in \mathfrak{gl}(d, \mathbb{R}) \) and the dynamical systems defined by them. As mentioned at the end of Section 3, we would like to characterize certain properties of \( A \) through these associated systems. We will study here, which properties of \( A \) are reflected by the induced dynamical systems on Grassmannian and flag manifolds. The reader will find all the details in Robinson [33], [15], and [7]. For background material on the projective space and Grassmannian and flag manifolds Boothby [8] is a good resource.

The **k-th Grassmannian** \( \mathbb{G}_k \) of \( \mathbb{R}^d \) can be defined via the following construction: Let \( F(k, d) \) be the set of \( k \)-frames in \( \mathbb{R}^d \), where a **k-frame** is an ordered set of \( k \) linearly independent vectors in \( \mathbb{R}^d \). Two \( k \)-frames \( X = \{x_1, \ldots, x_k\} \) and \( Y = \{y_1, \ldots, y_k\} \) are said to be equivalent, \( X \sim Y \), if there exists \( T \in \text{Gl}(k, \mathbb{R}) \) with \( X^T = TY^T \), where \( X \) and \( Y \) are interpreted as \( d \times k \) matrices. The quotient space \( \mathbb{G}_k = F(k, d)/\sim \) is a compact, \( (d-k) \)-dimensional differentiable manifold. For \( k = 1 \) we obtain the projective space \( \mathbb{P}^{d-1} = \mathbb{G}_1 \) in \( \mathbb{R}^d \).

The **k-th flag** of \( \mathbb{R}^d \) is given by the following \( k \)-sequences of subspace inclusions,

\[
F_k = \{ F_k = (V_1, \ldots, V_k), \; V_i \subset V_{i+1} \text{ and } \dim V_i = i \text{ for all } i \}.
\]

For \( k = d \) this is the **complete flag** \( F = F_d \).
A matrix $X \in \text{Gl}(d, \mathbb{R})$ (in particular, $X = e^{At}, t \in \mathbb{R}$) induces map on every Grassmannian $G_k$ and hence on the flags $F_k$ since the dimension of a subspace is invariant under $X$. For notational simplicity, we use the same symbol for these induced maps. In fact, more is true: A matrix $A \in \text{gl}(d, \mathbb{R})$ defines a map $G_A$ on the subspaces of $\mathbb{R}^d$ as follows: Let $V = \text{span}(\{x_1, \ldots, x_k\})$, then $G_A(V) = \text{span}(Ax_1, \ldots, Ax_k)$. This recipe defines a vector field on the Grassmannians $G_k$ and the flags $F_k$ of $\mathbb{R}^d$, and hence a (nonlinear) differential equation via $\dot{V} = G_A(V)$. We denote by $G_k \Phi$ and $F_k \Phi$ the induced dynamical systems on the Grassmannians and the flags, respectively.

Let us first look at the induced systems on the first Grassmannian $G_1$, i.e., on the projective space $\mathbb{P}^{d-1}$. This space of lines through the origin is obtained from the unit sphere $S^{d-1}$ by identifying opposite points. A metric is given by

$$d(F_v, F_{v'}) := \min \left( \frac{v}{\|v\|} - \frac{v'}{\|v'\|}, \frac{v}{\|v\|} + \frac{v'}{\|v'\|} \right). \quad (5.1)$$

The following lemma gives a concrete description of the corresponding differential equation.

**Lemma 5.1** For $A \in \text{gl}(d, \mathbb{R})$ let $\Phi$ be its linear flow in $\mathbb{R}^d$. The flow $\Phi$ projects onto a flow $P\Phi$ on $\mathbb{P}^{d-1}$, given by the differential equation

$$\dot{s} = h(s, A) = (A - s^T As) \; s, \text{ with } s \in \mathbb{P}^{d-1}.$$  

**Proof.** Exercise. ■

The characteristics of the projected flow $P\Phi$ are summarized in the following result. It also shows that the topological properties of this projected flow determine the decomposition of $\mathbb{R}^d$ into Lyapunov spaces.

**Theorem 5.2** Let $P\Phi$ be the projection onto $\mathbb{P}^{d-1}$ of a linear flow $\Phi(t, x) = e^{At}x$. Then the following assertions hold.

(i) $P\Phi$ has $l$ chain recurrent components $\{M_1, \ldots, M_l\}$, where $l$ is the number of different Lyapunov exponents (i.e. of different real parts of eigenvalues) of $A$.

(ii) For each Lyapunov exponent $\lambda_i$, $M_i = P\text{L}_i$, the projection of the $i$-th Lyapunov space onto $\mathbb{P}^{d-1}$. Furthermore $\{M_1, \ldots, M_l\}$ defines the finest Morse decomposition of $P\Phi$ and $M_i \prec M_j$ if and only if $\lambda_i < \lambda_j$.

(iii) For the sets in the finest Morse decomposition, the sets

$$\mathbb{P}^{-1}M_i := \{x \in \mathbb{R}^d, Px \in M_i\}$$

coincide with the Lyapunov spaces and hence yield a decomposition of $\mathbb{R}^d$ into linear subspaces

$$\mathbb{R}^d = \mathbb{P}^{-1}M_1 \oplus \ldots \oplus \mathbb{P}^{-1}M_l.$$ 

**Proof.** We may assume that $A$ is given in Jordan canonical form and we call the subspaces of $\mathbb{R}^d$ corresponding to a Jordan block Jordan subspaces. The solution formula shows that the Lyapunov spaces denoted by $L_j$ (recall Definition 2.12) yield a Morse decomposition of the flow $P\Phi$

$$\{P\text{L}_1, \ldots, P\text{L}_r\}.$$ 

Hence for assertions (i) and (ii) it remains to show that the flow $P\Phi$ restricted to projected Lyapunov space $P\text{L}_i$ is chain transitive. Then assertion (iii) is an immediate consequence of the fact that the $L_i$ are linear subspaces.

We may assume that the corresponding Lyapunov exponent is zero. This means that all real parts of the eigenvalues are zero. Consider first initial values in a Jordan subspace corresponding to a real eigenvalue, i.e., to the eigenvalue zero. The projective eigenvector $p_1$ (i.e. the eigenvector projected to $\mathbb{P}^{d-1}$) is an equilibrium for $P\Phi$. For all other initial values the projective solutions tend to $p$ for $t \to \pm \infty$, since they induce the highest polynomial growth in the component corresponding to the eigenvector. This shows that the projective Jordan subspace is chain transitive.
The sum of the Jordan subspaces corresponding to the eigenvalue zero is chain transitive, since the sums of eigenvectors yield a continuum of equilibria between the Jordan subspaces.

For a complex conjugate eigenvalue pair $\mu, \bar{\mu}$ i.e., to $\pm \imath \nu$, a real eigenvector $y_0 = (y_1, y_2, \ldots, 0)^T$ satisfies

$$y_1(t, y_0) = y_1 \cos \nu t - y_2 \sin \nu t, \quad y_2(t, y_0) = y_1 \sin \nu t + y_2 \cos \nu t$$

Thus it defines a $\frac{2\pi}{\nu}$-periodic solution in $\mathbb{P}^{d-1}$ and together they form a two-dimensional subspace of periodic solutions. As before, all projective solutions converge to this subspace as $t \to \pm \infty$ showing that the projective Jordan subspace is chain transitive. Similarly as above one shows that the sum of a Jordan corresponding to the eigenvalue zero and to a pair of complex conjugate eigenvalues $\pm \imath \nu$ yields a chain transitive component in projective space.

Now consider eigenvalue pairs $\pm \imath \nu_k$ for $k = 1, 2$, which are rationally dependent, i.e., there are $p_1, q_1, p_2, q_2 \in \mathbb{N}$ with

$$\frac{p_1}{q_1} \nu_1 = \frac{p_2}{q_2} \nu_2.$$ 

Then $p_1 q_2 \nu_1 + p_2 q_1 \nu_2 = 0$ and the corresponding eigensolutions as above have the (non-minimal) period $\frac{2\pi}{p_1 q_2 \nu_1} = \frac{2\pi}{p_2 q_1 \nu_2}$. Thus their linear combinations have the same period and, in projective space, they form a continuum of periodic solutions connecting the projective eigensolutions. Hence the corresponding projective Jordan subspaces are chain transitive.

It remains to discuss the case of eigenvalue pairs $\pm \imath \nu_k > 0$ for $k = 1, 2$, which are rationally independent. We show that for all points $x, y$ in the corresponding real eigenspaces and for all $T, \varepsilon > 0$ there is an $\varepsilon (x, T)$-chain from $x$ to $y$. By Proposition 4.11 it suffices to construct an $\varepsilon (x, 1)$-chain from $x$ to $y$ with jump times $T_i \in (1, 2)$. There is $\nu_2 \in \mathbb{R}$ arbitrarily close to $\nu_2$ such that $\nu_1$ and $\nu_2$ are rationally dependent. Hence we may choose $\tilde{\nu}_2$ such that the corresponding solutions for

$$A = \begin{pmatrix} 0 & -\imath \nu_1 & 0 & 0 \\ \imath \nu_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\imath \nu_2 \\ 0 & 0 & \imath \nu_2 & 0 \end{pmatrix}$$

and $\tilde{A} = \begin{pmatrix} 0 & -\imath \nu_1 & 0 & 0 \\ \imath \nu_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\imath \tilde{\nu}_2 \\ 0 & 0 & \imath \tilde{\nu}_2 & 0 \end{pmatrix}$

satisfy

$$d(\varphi(t, z), \tilde{\varphi}(t, z)) < \frac{\varepsilon}{2}$$

for all $z$ in projective space and all times $t \in [0, 2]$.

For $\tilde{A}$ there are $\varepsilon (\xi, T)$-chains from $x$ to $y$. Hence we find $\varepsilon (1, 1)$-chains from $x$ to $y$ for $A$ by cutting the trajectory pieces for $\tilde{A}$ into pieces of length $S_i \in (1, 2]$.

As above this shows that the sum of the corresponding Jordan blocks yields a chain transitive set in projective space.

Generalizing the whole discussion to sums of more than two eigenspaces and Jordan blocks one finally gets that the Lyapunov spaces yield chain transitive sets in projective space. ■

Theorem 5.2 shows that the direct sums of the generalized eigenspaces of $A \in gl(d, \mathbb{R})$ corresponding to eigenvalues with equal real part determine the finest Morse decomposition on the flow $\mathbb{P}\Phi$.

How do these Morse decompositions behave under equivalence of the flows on $\mathbb{P}^{d-1}$?

**Corollary 5.3** For $A, B \in gl(d, \mathbb{R})$ let $\mathbb{P}\Phi$ and $\mathbb{P}\Psi$ be the associated flows on $\mathbb{P}^{d-1}$ and suppose that there is a topological equivalence $h$ of $\mathbb{P}\Phi$ and $\mathbb{P}\Psi$. Then the chain recurrent components $N_1, \ldots, N_n$ of $\mathbb{P}\Psi$ are of the form $N_i = h [M_i]$, where $M_i$ is a chain recurrent component of $\mathbb{P}\Phi$. In particular the number of chain recurrent components of $\mathbb{P}\Phi$ and $\mathbb{P}\Psi$ agree, and $h$ maps the order on $\{M_1, \ldots, M_i\}$ onto the order on $\{N_1, \ldots, N_i\}$.

**Proof.** This is a consequence of Theorem 4.15. ■

But as it turns out, topological equivalences of 'linear' flows on the projective space $\mathbb{P}^{d-1}$ preserve much more than just the Lyapunov spaces.
Theorem 5.4 For $A, B \in gl(d, \mathbb{R})$ let $\mathbb{P} \Phi$ and $\mathbb{P} \Psi$ be the associated flows on $\mathbb{P}^{d-1}$ and suppose that there is a topological equivalence $h$ of $\mathbb{P} \Phi$ and $\mathbb{P} \Psi$. Then the projective subspaces corresponding to real Jordan blocks of $A$ are mapped onto projective subspaces corresponding to real Jordan blocks of $B$ preserving the dimensions. Furthermore, $h$ maps projective eigenspaces corresponding to real eigenvalues and to pairs of complex eigenvalues onto projective eigenspaces of the same type.

This result shows that while $C^0$-equivalence of projected linear flows on $\mathbb{P}^{d-1}$ determines the number $l$ of distinct Lyapunov exponents (i.e. real parts of the eigenvalues) of the matrix generators, it also characterizes the Jordan structure within each Lyapunov space (but, obviously, not the size of the Lyapunov exponents nor their sign). It imposes very restrictive conditions on the eigenvalues and the Jordan structure. Therefore, $C^0$-equivalences are not a useful tool to characterize $l$. The requirement of mapping orbits into orbits is too strong. A weakening leads us to the following characterization.

Theorem 5.5 Two matrices $A$ and $B$ in $gl(d, \mathbb{R})$ have the same list of dimensions $d_i$ of the Lyapunov spaces (in the natural order of their Lyapunov exponents) if and only if there exist a homeomorphism $h : \mathbb{P}^{d-1} \rightarrow \mathbb{P}^{d-1}$ that maps the finest Morse decomposition of $\mathbb{P} \Phi$ onto the finest Morse decomposition of $\mathbb{P} \Psi$, i.e., $h$ maps Morse sets onto Morse sets and preserves their orders.

Note that this theorem does not require a mapping of the trajectories, just of the finest Morse decomposition. Nevertheless, it characterizes the Lyapunov structure of a matrix $A \in gl(d, \mathbb{R})$.

Which additional information can be obtained from the induced dynamical systems on the higher order Grassmannians or the flag manifolds? This is a rather technical story and the reader not interested in some differential geometry may skip it - the time-varying theory presented in the next sections does not depend on the ideas discussed here. Actually, Remark 5.10 sums up the most interesting consequence, without being too specific.

Let us first look at the induced systems on the $k$-th flag.

Theorem 5.6 Let $A \in gl(d, \mathbb{R})$ with associated flows $\Phi$ on $\mathbb{R}^d$ and $\mathbb{P} \Phi$ on the $k$-flag.

(i) For every $k \in \{1, \ldots, d\}$ there exists a unique finest Morse decomposition $\{_{k,\mathcal{M}_{ij}}\}$ of $\mathcal{M} \Phi$, where $i_j \in \{1, \ldots, d\}$ is a multiindex, and the number of chain transitive components in $\mathcal{M} \Phi$ is bounded by $\frac{d^k}{(d-k)!}$.

(ii) Let $\mathcal{M}_i$ with $i \in \{1, \ldots, d\}$ be a chain recurrent component in $\mathcal{M}_{k-1}$. Consider the $(d-k+1)$-dimensional vector bundle $\pi : \mathcal{W}(\mathcal{M}_i) \rightarrow \mathcal{M}_i$ with fibers $\mathcal{W}(\mathcal{M}_i)_{\mathcal{M}_{k-1}} = \mathbb{R}^d / \mathcal{V}_{k-1}$ for $F_k = (V_1, \ldots, V_{k-1}) \in \mathcal{M}_i \subset \mathcal{F}_{k-1}$.

Then every chain recurrent component $\mu \mathcal{M}_{ij}$, $j = 1, \ldots, k_i \leq d - k + 1$, of the projective bundle $\mathbb{P} \mathcal{W}(\mathcal{M}_i)$ determines a chain recurrent component $k \mathcal{M}_{ij}$ on $\mathbb{P} \mathcal{F}$ via $k \mathcal{M}_{ij} = \{F_k = (F_{k-1}, V_k) \in \mathbb{P} \mathcal{F} : F_{k-1} \in \mathcal{M}_i \text{ and } \mathbb{P} (V_k / V_{k-1}) \in \mathcal{M}_{ij}\}$.

Every chain recurrent component in $\mathbb{P} \mathcal{F}$ is of this form - this determines the multiindex $i_j$ inductively for $k = 2, \ldots, d$.

For the Grassmannians we obtain as a consequence:

Corollary 5.7 On every Grassmannian $G_i$ there exists a finest Morse decomposition of the dynamical system $G_i \Phi$. Its Morse sets are given by the projection of the chain recurrent components from the complete flag $\mathbb{F}$.

Based on [15], the article [9] obtains a nice structure for the Morse sets on the Grassmannians:

Theorem 5.8 Let $A \in gl(d, \mathbb{R})$ be a matrix with flow $\Phi$ on $\mathbb{R}^d$. Let $L_i$, $i = 1, \ldots, l$, be the Lyapunov spaces of $A$, i.e., their projections $\mathbb{P} L_i = \mathcal{M}_i$ are the finest Morse decomposition of $P \Phi$ on the projective space. For $k = 1, \ldots, d$ define the index set $I(k) = \{(k_1, \ldots, k_m) : k_1 + \ldots + k_m = k \text{ and } 0 \leq k_i \leq d_i = \dim L_i\}$. 
Then the finest Morse decomposition on the Grassmannian $G_k$ is given by the sets

$$N_{k_1,\ldots,k_m}^k = G_{k_1}L_1 \oplus \ldots \oplus G_{k_m}L_m, \ (k_1,\ldots,k_m) \in I(k).$$

The following example shows how to compute, in specific cases, these Morse decompositions:

**Example 5.9** Consider the matrices

$$A = \text{diag}(-1,-1,1) \text{ and } B = \text{diag}(-1,1,1).$$

We obtain the following structure for the finest Morse decompositions of the flows $G_k\Phi$ on the Grassmannians for $A$:

- $G_1$: $M_1 = \text{span}(e_1,e_2)$ and $M_3 = \text{span}(e_3)$
- $G_2$: $M_{1,2} = \{\text{span}(e_1,e_2)\}$ and $M_{1,3} = \{\{\text{span}(x,e_3)\} : x \in \text{span}(e_1,e_2)\}$
- $G_3$: $M_{1,2,3} = \{\text{span}(e_1,e_2,e_3)\}$

and for $B$ we have

- $G_1$: $N_1 = \{\text{span}(e_1)\}$ and $N_2 = \{\text{span}(e_2,e_3)\}$
- $G_2$: $N_{1,2} = \{\text{span}(e_1,x) : x \in \text{span}(e_2,e_3)\}$ and $N_{2,3} = \{\text{span}(e_2,e_3)\}$
- $G_3$: $N_{1,2,3} = \{\text{span}(e_1,e_2,e_3)\}$.

On the other hand, the Morse sets of the flow $F\Phi$ in the full flag are given for $A$ and $B$ by

$$\begin{bmatrix} M_{1,2,3} \\ M_{1,2} \\ M_1 \end{bmatrix} \preceq \begin{bmatrix} M_{1,2,3} \\ M_{1,3} \\ M_1 \end{bmatrix} \leq \begin{bmatrix} M_{1,2,3} \\ M_{1,3} \\ M_3 \end{bmatrix} \text{ and } \begin{bmatrix} N_{1,2,3} \\ N_{1,2} \\ N_1 \end{bmatrix} \preceq \begin{bmatrix} N_{1,2,3} \\ N_{1,2} \\ N_3 \end{bmatrix} \preceq \begin{bmatrix} N_{1,2,3} \\ N_{1,2} \\ N_2 \end{bmatrix},$$

respectively. Thus in the full flag the numbers and the orders of the Morse sets coincide, while on the Grassmannians (together with the projection relations between different Grassmannians) one can distinguish also the dimensions of the corresponding Lyapunov spaces, see [7] for a precise statement.

**Remark 5.10** For two matrices $A, B \in \text{gl}(d,\mathbb{R})$ the lists of dimensions $d_i$ of the Lyapunov spaces (in the natural order of their Lyapunov exponents) are identical if and only if certain graphs defined on the Grassmannians are isomorphic, see [7].
6 Time-Varying Matrices and Lyapunov Exponents

Developing a linear algebra for time varying systems \( \dot{x} = A(t)x \) means defining appropriate concepts to generalize eigenvalues, linear eigenspaces and their dimensions, and certain normal forms that characterize the behavior of the solutions of a time varying system and that reduce to the constant matrix case if \( A(t) \equiv A \in \mathfrak{gl}(d, \mathbb{R}) \). As already observed by Lyapunov at the end of the nineteenth century, the eigenvalues and eigenspaces of the family \( \{ A(t), t \in \mathbb{R} \} \) do not provide the appropriate concept, see, e.g., Hahn [23], Chapter 62: Even if the real parts of all eigenvalues of all matrices \( A(t), t \in \mathbb{R} \) are negative, the origin \( 0 \in \mathbb{R}^d \) need not be a stable fixed point of \( \dot{x} = A(t)x \) (see Example 6.1). Instead one has to look at exponential growth rates of solution, called characteristic numbers or Lyapunov exponents. This section contains some examples elucidating the situation for general time varying systems \( \dot{x} = A(t)x \). It turns out that without further assumptions, no general theory can be developed. Hence in the ensuing sections, the framework of linear skew product flows is introduced and their properties are explored.

The announced example is due to Nemytskii and Vinograd [11], cp. G.A. Leonov [29].

Example 6.1 Consider the linear nonautonomous equation (with \( \pi \)-periodic coefficients)

\[ \dot{x} = A(t)x \text{ in } \mathbb{R}^2 \]

with

\[
A(t) = \begin{pmatrix}
1 - 4(\cos 2t)^2 & 2 + 2 \sin 4t \\
-2 + 2 \sin 4t & 1 - 4(\sin 2t)^2
\end{pmatrix}.
\]

Then one computes that the unbounded function (consider \( t = k\pi, k \in \mathbb{N} \))

\[
x(t) = \begin{pmatrix}
e^t \sin 2t \\
e^t \cos 2t
\end{pmatrix}
\]

is a solution (note that even \( \frac{1}{k\pi} \ln \|x(k\pi)\| \) is positive for \( k \to \infty \)). In fact

\[
\dot{x}(t) = \begin{pmatrix}
e^t \sin 2t + 2e^t \cos 2t \\
e^t \cos 2t - 2e^t \sin 2t
\end{pmatrix},
\]

and

\[
A(t)x(t) = \begin{pmatrix}
1 - 4(\cos 2t)^2 & 2 + 2 \sin 4t \\
-2 + 2 \sin 4t & 1 - 4(\sin 2t)^2
\end{pmatrix} \begin{pmatrix}
e^t \sin 2t \\
e^t \cos 2t
\end{pmatrix} = e^t \begin{pmatrix}
\sin 2t - 4(\cos 2t)^2 \sin t + 2 \cos 2t + 2 \sin 4t \cos 2t \\
-2 \sin 2t + 2 \sin 4t \sin t + \cos 2t - 4(\sin 2t)^2 \cos 2t
\end{pmatrix}.
\]

Using \( \sin 2x = 2 \sin x \cos x \), one finds, as claimed

\[
e^t \begin{pmatrix}
\sin 2t - 4(\cos 2t)^2 \sin t + 2 \cos 2t + 4 \sin 2t (\cos 2t)^2 \\
-2 \sin 2t + 4 (\sin 2t)^2 \cos 2t + \cos 2t - 4(\sin 2t)^2 \cos 2t
\end{pmatrix} = e^t \begin{pmatrix}
\sin 2t + 2 \cos 2t \\
-2 \sin 2t + \cos 2t
\end{pmatrix}.
\]
The eigenvalues $\lambda_1(t)$ and $\lambda_2(t)$ of the matrix $A(t)$ are equal to $-1$:

$$\det[A(t) - \lambda I] = \det\left(\begin{array}{cc} 1 - 4\cos 2t & 2 + 2\sin 4t \\ -2 + 2\sin 4t & 1 - 4\sin 2t \end{array}\right)$$

$$= (1 - 4\cos 2t)^2 - \lambda^2 (1 - 4\sin 2t)^2 - \lambda (-2 + 2\sin 4t) (2 + 2\sin 4t)$$

$$= (1 - 4\cos 2t)^2 (1 - 4\sin 2t)^2 - \lambda (1 - 4\cos 2t)^2 + 1 - 4\sin 2t)^2) + \lambda^2 - (-4 + 4\sin 2t)^2$$

$$= \lambda^2 + 2\lambda + (1 - 4\sin 2t)^2 - 4\cos 2t)^2 + 16\cos 2t)^2 + 4 - 16\sin 2t)^2 (\cos 2t)^2$$

$$= \lambda^2 + 2\lambda + 1.$$

This example shows that the analysis of linear non-autonomous differential equations via eigenvalues for fixed (or frozen) time $t$ is not a reasonable endeavour. (It should be remarked that, in particular, in the physics literature, this is not always acknowledged).

A classical notion due to A.M. Lyapunov specifies exponential growth rates of solutions to linear nonautonomous differential equations of the form

$$\dot{x}(t) = A(t)x(t),$$

where $t \rightarrow A(t) : \mathbb{R} \rightarrow gl(d, \mathbb{R})$. In order to ascertain solvability, we assume that this function is locally integrable. As usual, we denote (absolutely continuous) solutions with $x(t_0) = x_0 \in \mathbb{R}^d$ by $\varphi(t, t_0, x_0), t \in \mathbb{R}$. We omit the argument $t_0$, if $t_0 = 0$.

The Lyapunov exponent of $\varphi(t, t_0, x_0)$ is given by

$$\limsup_{t \to -\infty} \frac{1}{t} \ln \|\varphi(t, t_0, x_0)\|.$$

For Example 6.1, one immediately sees that the Lyapunov exponent of the solution (6.1) is equal to 1; however, the limit does not exist.

We note the following observations:

Instead of $\limsup$ one may also consider $\liminf$ or the corresponding limits for $t \to -\infty$.

The solutions are given using a fundamental solution $X(t, t_0)$ of $\dot{X}(t) = A(t)X(t)$ with $X(t_0, t_0) = I$ as

$$\varphi(t, t_0, x_0) = X(t + t_0, t_0)x_0.$$

Apparently, it is sufficient to consider solutions with initial time $t_0 = 0$, since for $t, s, t_0 \in \mathbb{R}$ the cocycle property

$$\varphi(t + s + t_0, s + t_0, \varphi(s + t_0, t_0, x_0)) = \varphi(t + s + t_0, t_0, x_0)$$

holds. Hence we define

$$\lambda(x_0) = \limsup_{t \to -\infty} \frac{1}{t} \ln \|\varphi(t, x_0)\|, \quad x_0 \in \mathbb{R}^d.$$

If $A(t) \equiv A$ is constant, the Lyapunov exponents are the real parts of the eigenvalues $\mu_i = \lambda_i + i\omega_i$ of $A$. Order the eigenvalues according to the real parts. Looking at the real Jordan canonical form, one finds that $\lambda(x_0) = \lambda_1$ iff $x_0$ has a component in a generalized eigenspace corresponding to an eigenvalue with real part $\lambda_1$ and no component in a generalized eigenspace corresponding to an eigenvalue with larger real part. Then the limit $\limsup$ is a limit. Similarly, also the limit for $t \to -\infty$ exists and its value is the smallest real part of an eigenvalue such that $x_0$ has a component in a corresponding generalized eigenspace.

The general nonautonomous case is much more complicated. See Cesari [12] for the following first results. The number of Lyapunov exponents is bounded by $d$. This follows, since solutions with different Lyapunov exponents are linearly independent. In fact, write

$$\lambda_p < \lambda_p - 1 < ... < \lambda_1$$

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for different values of Lyapunov exponents. We have to show that \( p \leq d \).

First note that \( \lambda(x + y) \leq \max\{\lambda(x), \lambda(y)\} \). The sets

\[
W_\lambda := \{x \in \mathbb{R}^d, \lambda(x) \leq \lambda \text{ or } x = 0\}
\]

are linear subspaces, since for \( \alpha \neq 0 \) and \( 0 \neq x, y \in \mathbb{R}^d \)

\[
\lambda(\alpha x) = \lambda(x) \text{ and } \lambda(x + y) \leq \max\{\lambda(x), \lambda(y)\}.
\]

These subspaces form a filtration, i.e.,

\[
\{0\} =: W_{p+1} \subset W_p \subset W_{p-1} \subset \ldots \subset W_1 \subset \mathbb{R}^d,
\]

and all inclusions are proper. Thus \( p \leq d \) follows.

Furthermore, on every base of \( \mathbb{R}^d \) the maximal Lyapunov exponent is attained. Suppose that \( \lambda(x_0) \) is maximal and let \( x_1, \ldots, x_d \) be a base. Then there are \( \alpha_1, \ldots, \alpha_d \in \mathbb{R} \) with

\[
x_0 = \alpha_1 x_1 + \cdots + \alpha_d x_d.
\]

This implies

\[
\varphi(t, x_0) = \varphi(t, \alpha_1 x_1 + \cdots + \alpha_d x_d) = \alpha_1 \varphi(t, x_1) + \cdots + \alpha_d \varphi(t, x_d).
\]

Hence

\[
\limsup_{t \to \infty} \frac{1}{t} \ln \|\varphi(t, x_0)\| \leq \max_{i=1, \ldots, d} \limsup_{t \to \infty} \frac{1}{t} \ln \alpha_i \|\varphi(t, x_i)\| = \max_{i=1, \ldots, d} \limsup_{t \to \infty} \frac{1}{t} \ln \|\varphi(t, x_i)\|
\]

as claimed. Here we have used that the exponential growth rate of a sum is bounded by the maximal exponential growth rate of the summands.

In general, no decomposition of \( \mathbb{R}^d \) into (time-dependent) subspaces, analogous to the Lyapunov decomposition for autonomous equations, exists.

## 7 Linear Skew-Product Flows

For certain classes of time varying systems it turns out that the Lyapunov exponents and Lyapunov spaces introduced in Section 6 capture the key properties of (real parts of) eigenvalues and of the associated subspace decomposition of \( \mathbb{R}^d \). These systems are linear skew product flows for which the base is a (nonlinear) system \( \theta_t \) that enters into the linear dynamics of a differential equation in the form \( \dot{x} = A(\theta_t)x \). Examples for this type of systems include periodic and almost periodic differential equations, random differential equations, systems over ergodic or chain recurrent bases, linear robust systems, and bilinear control systems. This section concentrates on periodic linear differential equations, random linear dynamical systems, and robust linear systems. It is written to emphasize the correspondences between the linear algebra in Section 2, Floquet theory, the multiplicative ergodic theorem, and the Morse spectrum and Selgrade’s theorem. This section is based on Arnold [3], Bronstein, Kopanskii [10], Colonius, Kliemann [14], Cong [17], and Robinson [33].

Lyapunov exponents for nonautonomous linear equations may not determine all desired stability properties, if no additional assumptions on the time dependency are made. A classical way is to assume that the differential equation is embedded in a family of differential equations and then to add topological or measure theoretic assumptions on the way the time dependency is generated. We follow this venue here.

We start by defining linear skew-product flows and their Lyapunov exponents.

**Definition 7.1** A (continuous time) linear skew-product flow is a dynamical system with state space \( M = B \times \mathbb{R}^d \) and flow \( \Phi : \mathbb{R} \times B \times \mathbb{R}^d \to B \times \mathbb{R}^d \), where \( \Phi = (\theta, \varphi) \) is defined as follows: \( \theta : \mathbb{R} \times B \to \Omega \) is a dynamical system, and \( \varphi : \mathbb{R} \times B \times \mathbb{R}^d \to \mathbb{R}^d \) is linear in its \( \mathbb{R}^d \)-component, i.e., for each \( (t, b) \in \mathbb{R} \times B \) the map \( \varphi(t, b, \cdot) : \mathbb{R}^d \to \mathbb{R}^d \) is linear. Skew-product flows are called measurable (continuous, differentiable) if \( B \) is a measurable space (topological space, differentiable manifold) and \( \Phi \) is measurable (continuous, differentiable). For the time-\( t \) maps, the notation \( \theta_t = \theta(t, \cdot) : B \to B \) is used again.
Note that the base component \( \theta : \mathbb{R} \times B \to B \) is a dynamical system itself, while the skew-component \( \varphi \) is not a dynamical system. The skew-component \( \varphi \) is often called a cocycle over \( \theta \).

**Definition 7.2** Let \( \Phi : \mathbb{R} \times B \times \mathbb{R}^d \to B \times \mathbb{R}^d \) be a linear skew-product flow. For \( x_0 \in \mathbb{R}^d \), \( x_0 \neq 0 \), the Lyapunov exponent is defined as \( \lambda(x_0, b) = \limsup_{t \to \infty} \frac{1}{t} \ln \| \varphi(t, b, x_0) \| \), where \( \ln \) denotes the natural logarithm and \( \| \cdot \| \) is any norm in \( \mathbb{R}^d \).

The following examples introduce the most important classes of linear skew product flows.

**Example 7.3** Time varying linear differential equations: Let \( A : \mathbb{R} \to \text{gl}(d, \mathbb{R}) \) be a uniformly continuous function and consider the linear differential equation \( \dot{x}(t) = A(t)x(t) \). The solutions of this differential equation define a dynamical system via \( \Phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}^d \), where \( \theta : \mathbb{R} \to \mathbb{R} \) is given by \( \theta(t, \tau) = t + \tau \), and \( \varphi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \) is defined as \( \varphi(t, \tau, x_0) = X(t, \tau)x_0 \). Here \( X(t, \tau) \) is a fundamental matrix of the differential equation \( X(t) = A(t)X(t) \) in \( \text{gl}(d, \mathbb{R}) \). Note that for \( \varphi(t, \tau, \cdot) : \mathbb{R}^d \to \mathbb{R}^d \), \( t \in \mathbb{R} \), we have \( \varphi(t+s, \tau) = \varphi(t, \tau(s, \cdot)) \circ \varphi(s, \tau) \) and hence the solutions of \( \dot{x}(t) = A(t)x(t) \) themselves do not define a flow. The additional component \( \theta \) keeps track of time.

**Example 7.4** Metric dynamical systems: Let \((\Omega, \mathcal{F}, P)\) be a probability space, i.e., a set \( \Omega \) with \( \sigma \)-algebra \( \mathcal{F} \) and probability measure \( P \). Let \( \theta : \mathbb{R} \times \Omega \to \Omega \) be a measurable flow such that the probability measure \( P \) is invariant under \( \theta \), i.e., \( \theta \) is \( \mathcal{F} \)-equivariant for all \( t \in \mathbb{R} \), or for all measurable sets \( X \in \mathcal{F} \) define \( \theta_tP(X) = P(\{\theta^{-1}_t(X)\}) = P(X) \). Flows of this form are often called metric dynamical systems.

**Example 7.5** Random linear dynamical systems: A random linear dynamical system is a skew-product flow \( \Phi : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \Omega \times \mathbb{R}^d \), where \((\Omega, \mathcal{F}, P, \theta)\) is a metric dynamical system and each \( \varphi : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d \) is linear in its \( \mathbb{R}^d \)-component. Examples for random linear dynamical systems are given, e.g., by linear stochastic differential equations or linear differential equations with stationary background noise, see [3].

**Example 7.6** Robust linear systems: Consider a linear system with time varying perturbations of the form \( \dot{x} = A(u(t))x := A_0x + \sum_{i=1}^m u_i(t)A_ix_0 \), where \( A_0, \ldots, A_m \in \text{gl}(d, \mathbb{R}) \), \( u \in U = \{u : \mathbb{R} \to U\) integrable on every bounded interval\} and \( U \subset \mathbb{R}^m \). A robust linear system defines a linear skew-product flow via the following construction: The base component is defined as the shift \( \theta : \mathbb{R} \times U \to U \), \( \theta(t, u) = u + t \), and the skew-component consists of the solutions \( \varphi(t, u(\cdot), x) \), \( t \in \mathbb{R} \) of the perturbed differential equation. Then \( \Phi : \mathbb{R} \times U \times \mathbb{R}^d \to U \times \mathbb{R}^d \), \( \Phi(t, u, x) = (\theta(t, u), \varphi(t, u, x)) \) defines a linear skew-product flow. The functions \( u \) can also be considered as (open loop) controls, compare [14]. Below we will discuss continuity properties of the flow.

A further example class is provided by differential equations linearized over an invariant set (we will come back to such systems in Part III.

**Example 7.7** Consider a differential equation

\[
\dot{y} = f(y), \quad y(0) = y_0 \in \mathbb{R}^d,
\]

where \( f : \mathbb{R}^d \to \mathbb{R}^d \) is \( C^1 \) and suppose that \( B \subset \mathbb{R}^d \) is an invariant set, i.e., the solutions \( \psi(t, y_0), t \in \mathbb{R} \), exist for all \( y_0 \in B \) and remain in \( B \), i.e., \( \psi(t, x_0) \in B \) for all \( t \in \mathbb{R} \). Denote the Jacobian of \( f \) along a trajectory \( \psi(t, y_0) \) by \( D_yf(\psi(t, y_0)) \) and consider the coupled system

\[
\dot{y} = f(y), \quad y(0) = y_0 \in \mathbb{R}^d,
\]

\[
\dot{x} = D_yf(\psi(t, y_0))x, \quad x(0) = x_0 \in \mathbb{R}^d, \quad (7.1)
\]

Then \( \Phi : \mathbb{R} \times B \times \mathbb{R}^d \to B \times \mathbb{R}^d \) is a skew product flow, where \( \Phi = (\theta, \varphi) \) is defined as follows: The base flow is \( \theta(t, y_0) = \psi(t, y_0) \) and \( \varphi(t, y_0, x_0) \) is the solution of the linear differential equation \( (7.2) \).
8 Periodic Linear Differential Equations - Floquet Theory

The general theory of time varying linear differential equations \( \dot{x}(t) = A(t)x(t) \) is still amazingly incomplete. Only for certain classes of functions \( A : \mathbb{R} \rightarrow \mathfrak{gl}(d, \mathbb{R}) \) do we have a satisfactory understanding of the qualitative behavior of the solutions. Historically the first complete theory for a class of time-varying linear systems was initiated by Floquet [18] in 1883 for the periodic case. In this section we briefly review Floquet’s theory and relate it to the idea of Lyapunov exponents and Lyapunov spaces as introduced in Section 2. Details supporting our discussion here can be found in Amann [2], Guckenheimer and Holmes [22], Hahn [23], Stoker [35], and Wiggins [37]. Partly, we follow the careful exposition in [13, Section 2.4].

**Definition 8.1** A periodic linear differential equation \( \dot{x} = A(t)x \) is given by a matrix function \( A : \mathbb{R} \rightarrow \mathfrak{gl}(d, \mathbb{R}) \) that is continuous and periodic (of period \( T > 0 \)). Similarly as in Example 7.3, we use the shift \( \theta(t, \tau) = t + \tau \mod T \). Then we may write \( \dot{x} = A(\theta(t, 0))x \) and the solutions define a dynamical system via \( \Phi : \mathbb{R} \times S^1 \times \mathbb{R}^d \rightarrow S^1 \times \mathbb{R}^d \), if we identify \( \mathbb{R} \mod T \) with the circle \( S^1 \).

Our first results concern the fundamental matrix of a periodic linear system.

We will need the following lemma which can be derived using the Jordan canonical form and the scalar logarithm (see e.g. Amann [2, Lemma 20.7] or Chicone [13, Theorem 2.47]). The difference between the real and the complex situation becomes already evident by looking at \(-1 = e^{i\pi}\).

**Lemma 8.2** For every invertible matrix \( S \in \mathfrak{gl}(d, \mathbb{C}) \) there is a matrix \( R \in \mathfrak{gl}(d, \mathbb{C}) \) such that \( S = e^R \). For every invertible matrix \( S \in \mathfrak{gl}(d, \mathbb{R}) \) there is a real matrix \( Q \in \mathfrak{gl}(d, \mathbb{R}) \) such that \( S^2 = e^Q \). The eigenvalues and Jordan blocks of \( R \) and \( Q \) are mapped onto the eigenvalues and eigenspaces of \( S \) and \( S^2 \), respectively.

**Proof.** Observe that in both cases it suffices to consider a (complex or real) Jordan block. For the first statement write \( S = \lambda I + N = \lambda(I + \frac{1}{\lambda} N) \) with nilpotent \( N \), i.e., \( N^m = 0 \) for some \( m \in \mathbb{N} \), and consider the series expansion for \( t \mapsto \ln(1 + t) \). Then \( S = e^R \) with

\[
R = (\ln \lambda) I + \sum_{j=1}^{m} \frac{(-1)^{j+1}}{j \lambda^j} N^j.
\]

For the second assertion observe that the real parts of the eigenvalues of \( S^2 \) are all positive. Then the real logarithm of these real parts exist and one can discuss the Jordan blocks similarly as above noting that a real logarithm of

\[
r \begin{pmatrix}
\cos \omega & -\sin \omega \\
\sin \omega & \cos \omega
\end{pmatrix}
\]
is \( \ln r I + \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \).

The proof above also shows that the Jordan blocks of \( R \) are mapped onto the Jordan blocks of \( e^R \) and the eigenvalues Similarly, the Jordan blocks of \( Q \) are mapped onto the Jordan blocks of \( S^2 = e^Q \).

**Remark 8.3** Another way to construct \( Q \) is to start with \( S = e^R \) and to define \( Q := R + \tilde{R} \in \mathfrak{gl}(d, \mathbb{R}) \). Then \( S^2 = e^R e^\tilde{R} = e^{R+\tilde{R}} = e^Q \).

**Remark 8.4** The real parts of the eigenvalues of \( R \) and \( Q \), respectively, are uniquely determined by \( S \). The imaginary parts are unique up to addition of \( 2k\pi i, k \in \mathbb{Z} \). In particular, several eigenvalues of \( R \) and \( Q \) may be mapped to the same eigenvalue of \( e^R \) and \( e^Q \), respectively.

The principal fundamental solution \( X(t), t \in \mathbb{R} \), is the unique solution of the matrix differential equation

\[
\dot{X}(t) = A(t)X(t) \text{ with initial value } X(0) = I.
\]

Then the solutions of \( \dot{x} = A(t)x \), \( x(0) = x_0 \), are given by \( x(t) = X(t)x_0 \).
**Theorem 8.5** Consider the periodic linear differential equation \( \dot{x} = A(t)x \) with period \( T > 0 \). The fundamental matrix \( X(t) \) of the system is of the form \( X(t) = P(t)e^{RT} \) for \( t \in \mathbb{R} \), where \( P(\cdot) \) is a non-singular, differentiable, and \( T \)-periodic matrix function and \( R \in \text{gl}(d, \mathbb{C}) \).

**Proof.** Let \( Y(t) := X(t + T)X(T)^{-1}, t \in \mathbb{R} \), and hence \( Y(0) = I \). Periodicity of \( A \) implies
\[
Y(t) = X(t + T)X(T)^{-1} = A(t)X(t + T)X(T)^{-1} = A(t + T)Y(t) = A(t)Y(t).
\]
By unique solvability of (8.1) this yields \( X(t) = Y(t) = X(t + T)X(T)^{-1} \) and hence \( X(t + T) = X(t)X(T) \) for all \( t \in \mathbb{R} \). By Lemma 8.2 there is a matrix \( R \in \text{gl}(d, \mathbb{C}) \) with \( X(T) = e^{TR} \). Now define a continuously differentiable matrix function \( P(t) \in GL(d, \mathbb{C}) \) by
\[
P(t) = X(t)e^{-tR}, \quad t \in \mathbb{R}.
\]
Then it follows that
\[
P(t + T) = X(t + T)e^{ -(t+T)R} = X(t)X(T)e^{-TR}e^{-tR} = X(t)e^{-tR} = P(t) \text{ for all } t \in \mathbb{R},
\]
showing that \( P(t) \) is \( T \)-periodic. \( \blacksquare \)

Theorem 8.5 indicates that the qualitative (stability) behavior of the solutions depends on the factor \( e^{RT} \). We list some properties and related notions for this factor.

**Remark 8.6** Consider the periodic linear differential equation \( \dot{x} = A(t)x \) with period \( T > 0 \). Let \( X(\cdot) \) be the fundamental solution with \( X(0) = I \in GL(d, \mathbb{R}) \). The matrix \( X(T) = e^{RT} \) is called the monodromy matrix of the system. Note that \( R \) is, in general, not uniquely determined by \( X \), and does not necessarily have real entries. The eigenvalues \( \alpha_j, j = 1, \ldots, d \) of \( X(T) \) are called the characteristic multipliers of the system. Observe that the spectrum of \( e^{RT} \) is given by \( e^{Re^{i\theta}} \). The eigenvalues \( \mu_j = \lambda_j + i\nu_j \) of \( R \) are the characteristic exponents. It holds that \( \mu_j = \frac{1}{T} \ln \alpha_j + \frac{2\pi m_i}{T}, \quad j = 1, \ldots, d \) and some \( m_i \in \mathbb{Z} \). This determines uniquely the real parts of the characteristic exponents \( \lambda_j = \text{Re } \mu_j = \ln |\alpha_j|, \quad j = 1, \ldots, d \). The \( \lambda_j \) are called the Floquet exponents of the system.

A nice application of Theorem 8.5 is the characterization of existence of periodic solutions:

**Corollary 8.7** Consider the \( T \)-periodic differential equation \( \dot{x} = A(t)x \). This equation has a non-trivial \( T \)-periodic solution in \( \mathbb{C}^d \) iff the system has a characteristic multiplier equal to 1. Then it also has a non-trivial \( 2T \)-periodic solution in \( \mathbb{R}^d \).

**Proof.** If \( x(t, x_0), t \in \mathbb{R} \), is a non-trivial \( T \)-periodic solution in \( \mathbb{C}^d \), then
\[
x_0 = X(T)x_0 \quad (8.2)
\]
and hence \( X(T) \) has a characteristic multiplier equal to 1. Conversely, if there is a characteristic multiplier equal to 1, there is 0 \( \neq x_0 \in \mathbb{C}^d \) satisfying (8.2). Then
\[
X(t + T)x_0 = X(t)X(T)x_0 = X(t)x_0, \quad t \in \mathbb{R}
\]
is a \( T \)-periodic solution in \( \mathbb{C}^d \). It remains to show that this yields existence of a \( 2T \)-periodic solution in \( \mathbb{R}^d \). For \( Q := \frac{1}{2}(R + \bar{R}) \in \text{gl}(d, \mathbb{R}) \) one obtains
\[
X(2T) = X(T)X(T) = e^{TR}e^{TR} = e^{T(R+R)} = e^{T(R+\bar{R})} = e^{2TQ}.
\]
Then
\[
z(t) := e^{tQ}(x_0 + \tilde{x}_0), \quad t \in \mathbb{R},
\]
is a non-trivial \( 2T \)-periodic solution in \( \mathbb{R}^d \), since, clearly, \( z(t) \in \mathbb{R}^d \) and, by (8.2), for all \( t \in \mathbb{R},
\[
z(t+2T) = e^{tQ}e^{2TQ}(x_0+\tilde{x}_0) = e^{tQ}X(2T)(x_0+\tilde{x}_0) = e^{tQ}[X(2T)x_0 + X(2T)\tilde{x}_0] = e^{tQ}[x_0 + \tilde{x}_0] = z(t).
\]
\( \blacksquare \)

Next we relate the Floquet exponents to the Lyapunov exponents \( \lambda(x_0) = \limsup_{t \to \infty} \frac{1}{t} \ln \| \varphi(t, x_0) \| \), where \( \varphi(t, x_0) \) denotes the solution of \( \dot{x} = A(t)x \) with \( \varphi(0, x_0) = x_0 \) (compare Definition 2.12).

We start with the following lemma on eigenvalues and eigenspaces.
Lemma 8.8 (i) The eigenvalues $\alpha_j$ of $X(T)$ are given by $e^{T\mu_j}$, where $\mu_j = \lambda_j + iv_j$ are the eigenvalues of $R$.

(ii) The eigenvalues of $X(2T) = X(T)^2$ are $e^{T\mu_j^Q}$, where $\mu_j^Q = \lambda_j^Q + iv_j^Q$ are the eigenvalues of $Q$.

(iii) The eigenvalues of $X(2T)$ are the squares of the eigenvalues of $X(T)$; in particular, the absolute values of the eigenvalues of $X(2T)$ are

$$|\alpha_j|^2 = e^{2T\lambda_j} = e^{T\lambda_j^Q}, \text{ and hence } \lambda_j = \frac{1}{2} \lambda_j^Q.$$ 

Proof. This is a consequence of Lemma 8.2 and the general relation $X(kT+s) = X(kT)$ for all $k \in \mathbb{Z}$ and $t \in \mathbb{R}$.

The following theorem for periodic linear differential equations is analogous to Theorem 2.13.

Theorem 8.9 Let $\Phi = (\theta, \varphi) : \mathbb{R} \times S^1 \times \mathbb{R}^d \rightarrow S^1 \times \mathbb{R}^d$ be the flow associated with a periodic linear differential equation $\dot{x} = A(t)x$. The system has a finite number of Lyapunov exponents $\lambda_j$, $j = 1, ..., l \leq d$. For each exponent $\lambda_j$ and each $\tau \in S^1$ there exists a splitting $\mathbb{R}^d = \bigoplus_{j=1}^l L(\lambda_j, \tau)$ of $\mathbb{R}^d$ into linear subspaces with the following properties:

(i) The subspaces $L(\lambda_j, \tau)$ have the same dimension independent of $\tau$, i.e. for each $j = 1, ..., l$ it holds that $\dim L(\lambda_j, \sigma) = \dim L(\lambda_j, \tau) =: d_i$ for all $\sigma, \tau \in S^1$.

(ii) The subspaces $L(\lambda_j, \tau)$ are invariant under the flow $\Phi$, i.e. for each $j = 1, ..., l$ it holds that $\varphi(t, \tau)L(\lambda_j, \tau) = L(\lambda_j, \varphi(t, \tau))$ for all $t \in \mathbb{R}$ and $\tau \in S^1$.

(iii) $\lambda(x, \tau) = \lim_{t \rightarrow \pm \infty} \frac{1}{t} \ln \|\varphi(t, \tau, x)\| = \lambda_j$ if and only if $x \in L(\lambda_j, \tau) \setminus \{0\}$.

Proof. By Lemma 8.8, the Floquet exponents $\lambda_j$ are given by $\lambda_j = \frac{1}{2} \lambda_j^Q$, where $\lambda_j^Q$ are the real parts of the eigenvalues of $Q \in gl(d, \mathbb{R})$.

First we show that the Floquet exponents are the Lyapunov exponents. By (8.4) we can write

$$X(kT+s) = X(kT)X(s) \text{ for all } k \in \mathbb{Z} \text{ and } t, s \in \mathbb{R}.$$ 

and recall that $X(2T) = e^Q$.

For the autonomous linear differential equation $\dot{y} = Qy$ Theorem 2.13 yields a decomposition of $\mathbb{R}^d$ into subspaces $L(\lambda_j)$ which are characterized by the property that the Lyapunov exponents for $t \rightarrow \pm \infty$ are given by the real parts $\lambda_j$ of the eigenvalues.

The continuously differentiable matrix function $P^Q(t) = X(t)e^{-Qt}$ maps the solution $e^{Qt}x_0$ of $\dot{x} = Qy, y(0) = x_0 \in \mathbb{R}^d$, to the solutions of $\dot{x} = A(t)x, x(0) = x_0$,

$$X(t)x_0 = X(t)e^{-Qt}e^{Qt}x_0 = P^Q(t)e^{Qt}x_0. \quad (8.5)$$

The exponential growth rates remain constant under multiplication by the bounded matrix $P^Q(t)$ with bounded inverse $P^Q(t)^{-1}$. Hence we get a corresponding decomposition of $\mathbb{R}^d$ which is characterized by the property that the exponential growth rates for a solution starting at time $t = 0$ in the corresponding subspace $L(\lambda_j, 0) : = L(\lambda_j)$ has exponential growth rate equal to a given Floquet exponent $\lambda_j$. Then

$$L(\lambda_j, \tau) : = X(\tau)L(\lambda_j, 0) \quad \tau \in \mathbb{R},$$

are subspaces which yields a splitting of $\mathbb{R}^d$ into subspaces characterized by the property that the exponential growth rates for a solution starting at time $t = \tau$ in the corresponding subspace $L(\lambda_j, \tau)$ has exponential growth rate equal to $\lambda_j$. But the exponential growth rate of the solution with $x(0) = x_0$ is equal to the exponential growth rate of the solution with $x(T) = x_0$. Hence the decomposition above is $T$-periodic and, clearly, it also depends continuously on $\tau$. ■

Combining Theorems 8.5 and 8.9 we obtain
Corollary 8.10 The Lyapunov exponents of the system are exactly the Floquet exponents. The linear subspaces \( L(\lambda_j, \cdot) \) are called the Lyapunov spaces (or sometimes the Floquet spaces) of the periodic matrix function \( A(t) \). For each \( j = 1, \ldots, l \leq d \) the map \( L_j : S^1 \rightarrow \mathbb{G}_{d_j} \) defined by \( \tau \mapsto L(\lambda_j, \tau) \) is continuous.

Proof. This follows from the construction of the spaces \( L(\lambda_j, \tau) \) and the corresponding properties of the Lyapunov spaces of the autonomous equation \( \dot{x} = Qx \).

These facts show that for periodic matrix functions \( A : \mathbb{R} \rightarrow gl(d, \mathbb{R}) \) the Floquet exponents and Floquet spaces replace the real parts of eigenvalues and the Lyapunov spaces, concepts that are so useful in the linear algebra of (constant) matrices \( A \in gl(d, \mathbb{R}) \). The number of Lyapunov exponents and the dimensions of the Lyapunov spaces are independent of \( \tau \in S^1 \), while the Lyapunov spaces themselves depend on the time parameter \( \tau \) of the periodic matrix function \( A(t) \), and they form periodic orbits in the Grassmannians \( \mathbb{G}_{d_j} \) and in the corresponding flag.

Remark 8.11 Transformations as \( Z(t) \) are known as Lyapunov transformations, see [23], Chapters 61-63.

Periodic linear differential equations yield periodic differential equations in projective space: As in Lemma 5.1, the flow \( \Phi : \mathbb{R} \times S^1 \times \mathbb{R}^d \rightarrow S^1 \times \mathbb{R}^d \) corresponding to \( \dot{x} = A(t)x \) projects onto a flow \( \mathbb{P}\Phi \) on \( S^1 \times \mathbb{P}^{d-1} \) where again the first component is the shift by \( \theta(t, \tau) = t + \tau \mod T \) and the second component is given by the solutions of the periodic differential equation
\[
\dot{s} = (A(t) - s^T A(t)s I) \, s \quad \text{with} \quad s \in \mathbb{P}^{d-1}.
\]

The next corollary characterizes the Lyapunov spaces for periodic linear differential equations by this projected flow. It is the analogue of Theorem 5.2.

Corollary 8.12 Let \( \mathbb{P}\Phi \) be the projection onto \( S^1 \times \mathbb{P}^{d-1} \) of a periodic linear flow as defined above. Then the following assertions hold.

(i) \( \mathbb{P}\Phi \) has \( l \) chain recurrent components \( \{ M_1, \ldots, M_l \} \), where \( l \) is the number of different Lyapunov exponents.

(ii) For each Lyapunov exponent \( \lambda_i \) one has that \( M_i = \{ (\tau, P_x) \mid x \in L_i(\lambda_i, \tau) \} \), the projection of the \( i \)-th Lyapunov space \( L_i(\lambda_i, \tau) \) onto \( \mathbb{P}^{d-1} \). Furthermore \( \{ M_1, \ldots, M_l \} \) defines the finest Morse decomposition of \( \mathbb{P}\Phi \) and \( M_i \prec M_j \) if and only if \( \lambda_i < \lambda_j \).

(iii) For the sets \( M_i \) in the finest Morse decomposition, the sets
\[
V_i^\tau := \{ x \in \mathbb{R}^d, (\tau, P_x) \in M_i \}, \quad \tau \in S^1,
\]
coincide with the Lyapunov subspaces and hence yield decompositions of \( \mathbb{R}^d \) into linear subspaces
\[
\mathbb{R}^d = V_1^\tau \oplus \ldots \oplus V_l^\tau, \quad \tau \in S^1.
\]

Proof. For the autonomous linear equation \( \dot{x} = Qx \) we have a decomposition of \( \mathbb{R}^d \) into the Lyapunov spaces \( L(\lambda_j, 0) \) which by Theorem 5.2 correspond to the Morse sets in the finest Morse decomposition. By (8.5) the matrix function \( P^d(t) \) maps the solution of \( \dot{x} = Qy, y(0) = x_0 \in \mathbb{R}^d \), to the solution of \( \dot{x} = A(t)x, x(0) = x_0 \). Since these maps and their inverses are uniformly bounded by compactness of \( S^1 \times \mathbb{P}^{d-1} \), one can show that the maximal chain transitive sets in \( \mathbb{P}^{d-1} \) are mapped onto the maximal chain transitive sets in \( S^1 \times \mathbb{P}^{d-1} \). Then the assertions follow.

As an application of these results, consider the problem of stability of the zero solution of \( \dot{x}(t) = A(t)x(t) \) with period \( T > 0 \). The following definition generalizes the last part of Definition 2.12.

Definition 8.13 The stable, center, and unstable subspaces associated with the periodic matrix function \( A : \mathbb{R} \rightarrow gl(d, \mathbb{R}) \) are defined as \( L^- (\tau) = \bigoplus \{ L(\lambda_j, \tau) \mid \lambda_j > 0 \} \), \( L^0 (\tau) = \bigoplus \{ L(\lambda_j, \tau) \mid \lambda_j = 0 \} \), and \( L^+ (\tau) = \bigoplus \{ L(\lambda_j, \tau) \mid \lambda_j > 0 \} \), respectively, for \( \tau \in S^1 \).

With these preparations we can state the main result regarding stability of periodic linear differential equations.
Theorem 8.14 The zero solution \( x(t, 0) \equiv 0 \) of the periodic linear differential equation \( \dot{z} = A(t)z \)

is asymptotically stable if and only if it is exponentially stable if and only if all Lyapunov exponents

are negative if and only if \( L^-(\tau) = \mathbb{R}^d \) for some (and hence for all) \( \tau \in \mathbb{S}^1 \).

To show the power of Floquet’s approach we discuss two classical examples.

Example 8.15 Hamiltonian systems: Let \( H \) be a continuous quadratic form in \( 2d \) variables \( x_1, \ldots, x_d, y_1, \ldots, y_d \) and consider the Hamiltonian system

\[
\dot{x}_i = \frac{\partial H}{\partial y_i}, \quad \dot{y}_i = -\frac{\partial H}{\partial x_i}, \quad i = 1, \ldots, d.
\]

Using \( z^T = [x^T, y^T] \) we can set \( H(x, y, t) = z^T A(t) z \), where

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}
\]

symmetric, and hence the equation takes the form

\[
\dot{z} = \begin{bmatrix} A_{12}(t) & A_{22}(t) \\ -A_{11}(t) & -A_{12}(t) \end{bmatrix} z = P(t) z.
\]

Note that \( -P^T(t) = QP(t)Q^{-1} \) with \( Q = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \) where \( I \) is the \( d \times d \) identity matrix. Assume that \( H \) is \( T \)-periodic, then the equation for \( z \) and its adjoint have the same Floquet exponents and for each exponent \( \lambda \) its negative \( -\lambda \) is also a Floquet exponent. Hence the fixed point \( 0 \in \mathbb{R}^{2d} \) cannot be exponentially stable, compare [23], Chapter 60.

Example 8.16 Hill-Mathieu equations: Consider the periodic linear oscillator

\[
\ddot{y} + q_1(t)\dot{y} + q_2(t)y = 0.
\]

Using the substitution \( y = z \exp(-\frac{1}{2}\int q_1(u)du) \) one obtains Hill’s differential equation

\[
\ddot{z} + p(t)z = 0, \quad p(t) := q_2(t) - \frac{1}{4}q_1(t)^2 - \frac{1}{2}\dot{q}_1(t).
\]

Its characteristic equation is \( \lambda^2 - 2a\lambda + 1 = 0 \), with a still to be determined. The multipliers satisfy the relations \( \alpha_1\alpha_2 = 1 \) and \( \alpha_1 + \alpha_2 = 2a \). The exponential stability of the system can be analyzed using the parameter \( a \): If \( a^2 > 1 \), then one of the multipliers has absolute value \( > 1 \), and hence the system has an unbounded solution. If \( a^2 = 1 \), then the system has a non-trivial periodic solution according to Example 1. If \( a^2 < 1 \), then the system is stable. The parameter \( a \) can often be expressed in form of a power series, see [23], Chapter 62, for more details. A special case of Hill’s equation is the Mathieu equation

\[
\ddot{z} + (\beta_1 + \beta_2 \cos 2t)z = 0,
\]

with \( \beta_1, \beta_2 \) real parameters. For this equation numerically computed stability diagrams are available, see, e.g., [35], Chapters VI.3 and 4.

9 Random Linear Dynamical Systems

In this section we consider time-varying matrix functions \( A(\theta(t, \omega)) \) that depend on an underlying stochastic process \( \theta(t, \omega) \). The ‘linear algebra’ for these matrices was developed by Oseledec in [32], using Lyapunov exponents. We will give a brief overview of his theory and some subsequent developments. We refer to [3] and [17] for the details. A reader who is not familiar with the basics of measure and probability theory may skip to the next section.

As in Examples 7.4 and 7.5 we denote by \( \theta : \mathbb{R} \times \Omega \to \Omega \) a metric dynamical system and by \( \Phi : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \Omega \times \mathbb{R}^d \) a random linear dynamical system. We recall the definitions of invariant measures and ergodicity in this context:
Definition 9.1 Let $\theta : \mathbb{R} \times \Omega \rightarrow \Omega$ be a metric dynamical system on the probability space $(\Omega, \mathcal{F}, P)$. A set $\Delta \in \mathcal{F}$ is called $P$-invariant under $\theta$ if $P\left(\left[\theta^{-1}(t, \Delta) \setminus \Delta\right] \cup \left[\Delta \setminus \theta^{-1}(t, \Delta)\right]\right) = 0$ for all $t \in \mathbb{R}$. The flow $\theta$ is called ergodic, if each invariant set $\Delta \in \mathcal{F}$ has $P$-measure 0 or 1.

As an example of a metric dynamical system we recall Kolmogorov’s construction:

Example 9.2 Let $(\Gamma, \mathcal{E}, Q)$ be a probability space and $\xi : \mathbb{R} \times \Gamma \rightarrow \mathbb{R}^m$ a stochastic process with continuous trajectories, i.e. the functions $\xi(\cdot, \gamma) : \mathbb{R} \rightarrow \mathbb{R}^m$ are continuous for all $\gamma \in \Gamma$. The process $\xi$ can be written as a measurable dynamical system in the following way: Define $\Omega = C(\mathbb{R}, \mathbb{R}^m)$, the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}^m$. We denote by $\mathcal{F}$ the $\sigma$-algebra on $\Omega$ generated by the cylinder sets, i.e. by sets of the form $Z = \{\omega \in \Omega, \omega(t_1 \in F_1, ..., \omega(t_n) \in F_n, n \in \mathbb{N}, F_i \text{ Borel sets in } \mathbb{R}^m\}$. The process $\xi$ induces a probability measure $P$ on $(\Omega, \mathcal{F})$ via $P(Z) = Q\{\gamma \in \Gamma, \xi(t_1, \gamma) \in F_1, \text{ for } i = 1, ..., n\}$. Define the shift $\theta : \mathbb{R} \times \Omega \rightarrow \mathbb{R} \times \Omega$ as $\theta(t, \omega) = (t, \omega(t^+))$. Then $(\Omega, \mathcal{F}, P, \theta)$ is a measurable dynamical system. If $\theta$ is stationary, i.e. if for all $n \in \mathbb{N}$, and $t, t_1, ..., t_n \in \mathbb{R}$ and all Borel sets $F_1, ..., F_n$ in $\mathbb{R}^m$, then it holds that $Q\{\gamma \in \Gamma, \xi(t_1, \gamma) \in F_1 \text{ for } i = 1, ..., n\} = Q\{\gamma \in \Gamma, \xi(t_1 + t, \gamma) \in F_i \text{ for } i = 1, ..., n\}$, then the shift $\theta$ on $\Omega$ is $P$-invariant, and $(\Omega, \mathcal{F}, P, \theta)$ is a metric dynamical system.

As an application of these ideas, we consider random linear differential equations:

Example 9.3 Let $A : \Omega \rightarrow \text{gl}(d, \mathbb{R})$ be measurable with $A \in L^1$. Consider the random linear differential equation $\dot{x}(t) = A(\theta(t, \omega))x(t)$ where $(\Omega, \mathcal{F}, P, \theta)$ is a metric dynamical system as described before. We understand the solutions of this equation to be $\omega$-wise. Then the solutions define a random linear dynamical system.

The key theorem in this section is Oseledets’ Multiplicative Ergodic Theorem that generalizes the results for constant and periodic matrix functions to the random context:

Theorem 9.4 Consider a random linear dynamical system $\Phi = (\theta, \varphi) : \mathbb{R} \times \Omega \times \mathbb{R}^d \rightarrow \Omega \times \mathbb{R}^d$ and assume
\[
\sup_{0 \leq t \leq 1} \ln^+ \|\varphi(t, \omega)\| \in L^1(\Omega, \mathcal{F}, P) \quad \text{and} \quad \sup_{0 \leq t \leq 1} \ln^+ \|\varphi(t, \omega)^{-1}\| \in L^1(\Omega, \mathcal{F}, P),
\]
where $\| \cdot \|$ is any norm on $\text{GL}(d, \mathbb{R})$, $L^1$ is the space of integrable functions, and $\ln^+$ denotes the positive part of $\ln$, i.e.,
\[
\ln^+(x) = \begin{cases} \ln(x) & \text{for } \ln(x) > 0 \\ 0 & \text{for } \ln(x) \leq 0. \end{cases}
\]
Then there exists a set $\widehat{\Omega} \subset \Omega$ of full $P$-measure, invariant under the flow $\theta : \mathbb{R} \times \Omega \rightarrow \Omega$, such that for each $\omega \in \widehat{\Omega}$ there is a splitting $\mathbb{R}^d = \bigoplus_{j=1}^{l(\omega)} L_j(\omega)$ of $\mathbb{R}^d$ into linear subspaces with the following properties:

(i) The number of subspaces is $\theta$-invariant, i.e. $l(\theta(t, \omega)) = l(\omega)$ for all $t \in \mathbb{R}$, and the dimensions of the subspaces are $\theta$-invariant, i.e. $\dim L_j(\theta(t, \omega)) = \dim L_j(\omega) =: d_j(\omega)$ for all $t \in \mathbb{R}$.

(ii) The subspaces are invariant under the flow $\Phi$, i.e. $\varphi(t, \omega)L_j(\omega) \subset L_j(\theta(t, \omega))$ for all $j = 1, ..., l(\omega)$.

(iii) There exist finitely many numbers $\lambda_1(\omega) < ... < \lambda_{l(\omega)}(\omega)$ in $\mathbb{R}$ (with possibly $\lambda_1(\omega) = -\infty$), such that for each $x \in \mathbb{R}^d \setminus \{0\}$ the Lyapunov exponent $\lambda(x, \omega)$ exists as a limit and $\lambda(x, \omega) = \lim_{t \rightarrow +\infty} \frac{1}{t} \ln \|\varphi(t, \tau, x)\| = \lambda_j(\omega)$ if and only if $x \in L_j(\omega) \setminus \{0\}$.

The subspaces $L_j(\omega)$ are called the Lyapunov (or sometimes the Oseledets) spaces of the system $\Phi$.

The Lyapunov exponents and Lyapunov spaces in Theorem 9.4 satisfy the following measurability properties:

Corollary 9.5 The following maps are measurable: $l : \Omega \rightarrow \{1, ..., d\}$ with the discrete $\sigma$-algebra, and for each $j = 1, ..., l(\omega)$ the maps $L_j : \Omega \rightarrow \mathcal{A}_d$ with the Borel $\sigma$-algebra, $d_j : \Omega \rightarrow \{1, ..., d\}$ with the discrete $\sigma$-algebra, and $\lambda_j : \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$ with the (extended) Borel $\sigma$-algebra.
If the underlying metric dynamical systems, the base flow, is ergodic, some of the objects in the Multiplicative Ergodic Theorem do not depend on $\omega$:

**Corollary 9.6** If the base flow $\theta : \mathbb{R} \times \Omega \longrightarrow \Omega$ is ergodic, then the maps $l$, $d_j$, and $\lambda_j$ are constant on $\tilde{\Omega}$, but the Lyapunov spaces $L_j(\omega)$ still depend (in a measurable way) on $\omega \in \tilde{\Omega}$.

To understand the importance of Oseledets’ theorem we apply it to the previously discussed cases of constant and periodic matrix functions:

**Example 9.7** The case of constant matrices: Let $A \in \text{gl}(d, \mathbb{R})$ and consider the dynamical system $\varphi : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ generated by the solutions of the linear differential equation $\dot{x} = Ax$. The flow $\varphi$ can be considered as the skew-component of a random linear dynamical system over the base flow given by $\Omega = \{0\}$, $\mathcal{F}$ the trivial $\sigma$-algebra, $P$ the Dirac measure at $\{0\}$, and $\theta : \mathbb{R} \times \Omega \longrightarrow \Omega$ defined as the constant map $\theta(t, \omega) = \omega$ for all $t \in \mathbb{R}$. Since the flow is ergodic and satisfies the integrability condition, we can recover all the results on Lyapunov exponents and Lyapunov spaces for $\varphi$ from the Multiplicative Ergodic Theorem.

**Example 9.8** Weak Floquet theory: Let $A : \mathbb{R} \longrightarrow \text{gl}(d, \mathbb{R})$ be a continuous, periodic matrix function. Define the base flow as follows: $\Omega = \mathbb{S}^1$, $\mathcal{B}$ is the Borel $\sigma$-algebra on $\mathbb{S}^1$, $P$ is the uniform distribution on $\mathbb{S}^1$, and $\theta$ is the shift $\theta(t, \tau) = t + \tau$. Then $(\Omega, \mathcal{F}, P, \theta)$ is an ergodic metric dynamical system. The solutions $\varphi(\cdot, \tau, x)$ of $\dot{x} = A(t)x$ define a random linear dynamical system $\Phi : \mathbb{R} \times \Omega \times \mathbb{R}^d \longrightarrow \Omega \times \mathbb{R}^d$ via $\Phi(t, \omega, x) = (\theta(t, \omega), \varphi(t, \omega, x))$. With this set-up, the Multiplicative Ergodic Theorem recovers the results of Floquet Theory with $P$-probability 1.

For random linear differential equations as in Example 9.3, Theorem 9.4 yields the following:

**Example 9.9** Continuation of Example 9.3: Since we assume that $A \in \mathcal{L}^1$, this system satisfies the integrability conditions of the Multiplicative Ergodic Theorem. Hence for random linear differential equations $\dot{x}(t) = A(\theta(t, \omega))x(t)$ the Lyapunov exponents and the associated Oseledets spaces replace the real parts of eigenvalues and the Lyapunov spaces of constant matrices $A \in \text{gl}(d, \mathbb{R})$. If the ‘background’ process $\theta$ is ergodic, then all the quantities in the Multiplicative Ergodic Theorem are constant, except for the Lyapunov spaces that do, in general, depend on chance.

The problem of stability of the zero solution of $\dot{x}(t) = A(\theta(t, \omega))x(t)$ can now be analyzed in analogy to the case of a constant matrix or a periodic matrix function: The **stable, center, and unstable subspaces** associated with the random matrix process $A(\theta(t, \omega))$ are defined as $L^-(\omega) = \bigoplus\{L_j(\omega), \lambda_j(\omega) < 0\}$, $L^0(\omega) = \bigoplus\{L_j(\omega), \lambda_j(\omega) = 0\}$, and $L^+(\omega) = \bigoplus\{L_j(\omega), \lambda_j(\omega) > 0\}$, respectively for $\omega \in \tilde{\Omega}$. We obtain the following characterization of stability:

**Theorem 9.10** The zero solution $x(t, \omega, 0) \equiv 0$ of the random linear differential equation $\dot{x}(t) = A(\theta(t, \omega))x(t)$ is $P$-almost surely exponentially stable if and only if $P$-almost surely all Lyapunov exponents are negative if and only if $P\{\omega \in \tilde{\Omega}, L^-(\omega) = \mathbb{R}^d\} = 1$.

This theorem goes a long way in characterizing stability for systems defined by random matrix functions $A(\theta(t, \omega))$: It determines the almost sure stability behavior. One could, of course, also look at the $p$-th moments, $p \geq 0$, of the trajectories and try to determine their stability behavior. Results in this direction were first obtained in [4] for Markovian background noise $\theta(t, \omega)$, compare [3] for a complete overview.

In general, Lyapunov exponents for random linear systems are difficult to compute explicitly - numerical methods are usually the way to go, see e.g. [36], [20], or the engineering oriented monograph [38]. But average Lyapunov exponents for random linear differential equations can sometimes be computed explicitly. In the ergodic case, the average Lyapunov exponent $\bar{\lambda} := \frac{1}{2} \sum d_j \lambda_j$ is given by $\bar{\lambda} = \frac{1}{2} \text{tr} \mathbb{E}(A | \mathcal{I})$, where $A : \Omega \longrightarrow \text{gl}(d, \mathbb{R})$ is the random matrix of the system, and $\mathbb{E}(\cdot | \mathcal{I})$ is the conditional expectation of the probability measure $P$ given the $\sigma$-algebra $\mathcal{I}$ of invariant sets on $\Omega$. 

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Example 9.11 **Average Lyapunov exponent:** As an example, consider the linear oscillator with random restoring force
\[ \ddot{y}(t) + 2\beta \dot{y}(t) + (1 + \sigma f(\theta(t, \omega)))y(t) = 0, \]
where \( \beta, \sigma \in \mathbb{R} \) are positive constants and \( f : \Omega \to \mathbb{R} \) is in \( L^1 \). We assume that the background process is ergodic. Using the notation \( x_1 = y \) and \( x_2 = \dot{y} \) we can write the equation as
\[ \dot{x}(t) = A(\theta(t, \omega))x(t) = \begin{pmatrix} 0 & 1 \\ -1 - \sigma f(\theta(t, \omega)) & -2\beta \end{pmatrix} x(t). \]
For this system we obtain \( \lambda = -\beta \), compare [3], Remark 3.3.12.

## 10 Robust Linear Systems

In this section we consider a third class of time-varying matrices, those that define robust (or controlled) linear systems as in Example 7.6. First we show that they define continuous linear skew product flows.

Consider
\[ \dot{x} = A(u(t))x := A_0 x + \sum_{i=1}^{m} u_i(t)A_i x, \tag{10.1} \]
where \( A_0, ..., A_m \in gl(d, \mathbb{R}) \), \( u \in \mathcal{U} = \{ u : \mathbb{R} \to U, \text{integrable on every bounded interval} \} \) and \( U \subset \mathbb{R}^m \) compact and convex. The base component is defined as the shift \( \theta : \mathbb{R} \times \mathcal{U} \to U \), \( \theta(t, u(\cdot)) = u(\cdot + t) \), and the skew-component consists of the solutions \( \varphi(t, u(\cdot), x) \), \( t \in \mathbb{R} \) of the perturbed differential equation.

We cite the following result ([14, Lemma 4.2.2]) which follows from standard functional analysis.

**Lemma 10.1** Let \( U \subset \mathbb{R}^m \), \( m \geq 1 \), be a compact and convex set and consider the set
\[ \mathcal{U} = \{ u : \mathbb{R} \to U, \text{measurable} \} \subset L_\infty(\mathbb{R}, \mathbb{R}^m) \]
The set \( \mathcal{U} \) is compact and metrizable in the weak* topology of \( L_\infty(\mathbb{R}, \mathbb{R}^m) = (L_1(\mathbb{R}, \mathbb{R}^m))^* \); a metric is given by
\[ d(u, v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left| \frac{1}{1 + \int_{\mathbb{R}} |u(t) - v(t), x_n(t)| \, dt} \right|, \tag{10.2} \]
where \( \{ x_n, n \in \mathbb{N} \} \) is a countable, dense subset of \( L_1(\mathbb{R}, \mathbb{R}^m) \), and \( \langle \cdot, \cdot \rangle \) denotes an inner product in \( \mathbb{R}^m \). With this metric, \( \mathcal{U} \) is a compact, complete, separable metric space.

From now on we will consider \( \mathcal{U} \) as a metric space with a fixed metric given by (10.2). We now turn to the analysis of the shift space \((\mathcal{U}, \theta)\). Note that \( u \in \mathcal{U} \) is a periodic function if and only if \( u \) is a periodic point of the flow \((\mathcal{U}, \theta)\).

**Lemma 10.2** The shift \( \theta \) defines a continuous dynamical system on \( \mathcal{U} \). The periodic functions are dense in \( \mathcal{U} \) and hence the shift on \( \mathcal{U} \) is chain transitive.

**Proof.** Obviously \( \theta_{t+s} = \theta_t \circ \theta_s \) for \( t, s \in \mathbb{R} \) and \( \theta_0 = \text{id} \). In order to prove continuity of \( \theta \) we consider sequences \( t_n \to t \) in \( \mathbb{R} \) and \( u_n \to u \) in \( \mathcal{U} \). Then for all \( x \in L_1(\mathbb{R}, \mathbb{R}^m) \)
\[
\left| \int_{\mathbb{R}} \langle u_n(t + \tau), x(\tau) \rangle \, d\tau - \int_{\mathbb{R}} \langle u(t + \tau), x(\tau) \rangle \, d\tau \right|
\leq \left| \int_{\mathbb{R}} \langle u_n(t + \tau) - u_n(t + \tau), x(\tau) \rangle \, d\tau \right| + \left| \int_{\mathbb{R}} \langle u_n(t + \tau) - u(t + \tau), x(\tau) \rangle \, d\tau \right|
= \left| \int_{\mathbb{R}} \langle u_n(\tau), x(\tau - t_n) \rangle \, d\tau - \int_{\mathbb{R}} \langle u_n(\tau), x(\tau - t) \rangle \, d\tau \right| + \left| \int_{\mathbb{R}} \langle u_n(\tau) - u(\tau), x(\tau - t) \rangle \, d\tau \right|. \]
The second summand converges to zero because $u_n \to u$ in $U$; the first can be estimated by

$$\leq \sup_{w \in U} |w| \int_{\mathbb{R}} |x(\tau - t_n) - x(\tau - t)| \, dr.$$  

Here the integral converges to zero as $t_n \to t$. This well-known fact ("continuity of the norm in $L_1$") can be seen as follows: For a bounded interval $I \subset \mathbb{R}$ and $t \in \mathbb{R}$ define $I(t) = I + t$. Then the characteristic function $\chi_I(\tau) := 1$ for $\tau \in I$ and $\chi_I(\tau) := 0$ elsewhere, satisfies

$$\int_{I} |\chi_I(\tau - t) - \chi_I(\tau)| \, d\tau = \int_{I(t)} d\tau + \int_{I(t) \setminus I} d\tau = \lambda(I) + \lambda(I(t)) - 2\lambda(I \cap I(t)).$$

Let $|t_n| \to 0$ with $I(t_1) \cap I \subset I(t_2) \cap I \subset \ldots \subset I(t_n) \cap I \subset \ldots$. Then $\bigcup_{n=1}^{\infty} I(t_n) \cap I = I$ and hence we obtain for the Lebesgue measures $\lim_{n \to \infty} m(I \cap I(t_n)) = m(I) = \lim_{n \to \infty} m(I(t_n))$. This proves the assertion for $\chi_I$ at $t = 0$. The assertion for arbitrary $t \in \mathbb{R}$, for step functions, and then for all elements $x \in L_1$ follows in a standard way. We conclude that $\theta(t_n, u_n) = u_n(t_n + \cdot) \to \theta(t, u) = u(t + \cdot)$ in $U$.

Density of periodic functions follows using similar arguments ([14, Lemma 4.2.2].

**Proposition 10.3** The perturbed linear system (10.1) defines a continuous linear skew product flow $\Phi = (\theta, \varphi)$ on $U \times \mathbb{R}^d$.

**Proof.** By Lemma 10.2 the shift $\theta$ on $U$ is continuous. The group properties are clearly satisfied and the shift $\Theta$ is continuous by Lemma 10.2. For continuity of $\Phi$ consider sequences $t^n \to t^0$ in $\mathbb{R}$, $u^n \to u^0$ in $U$, and $x^n \to x^0$ in $\mathbb{R}^d$. Abbreviate $\varphi^n(t) = \varphi(t, x^n, u^n)$, $t \in \mathbb{R}$, $n = 0, 1, \ldots$. We have to show that $\varphi^n(t^n) \to \varphi^0(t^0)$. One finds

$$||\varphi^n(t^n) - \varphi^0(t^0)|| \leq ||\varphi^n(t^n) - \varphi^n(t^0)|| + ||\varphi^n(t^0) - \varphi^0(t^0)||.$$  

The first summand tends to zero by uniform boundedness of the derivatives and

$$||\varphi^n(t^0) - \varphi^0(t^0)||$$

$$\leq ||x^n - x^0|| + \left|\int_{0}^{t^0} A_0 [\varphi^n(\tau) - \varphi^0(\tau)] \, d\tau\right| + \left|\int_{0}^{t^0} \sum_{i=1}^{m} u^n_i(\tau)A_i [\varphi^n(\tau) - \varphi^0(\tau)] \, d\tau\right|$$

$$+ \left|\int_{0}^{t^0} \sum_{i=1}^{m} [u^n_i(\tau) - u^n_i(t)] A_i \varphi^0(\tau) \, d\tau\right|.$$  

Here the first and fourth summands converge to zero by assumption; the second and third are bounded from above by $[\max_i \|A_i\| + \max_{u \in U} \|u\|] \int_{0}^{t^0} ||\varphi^n(\tau) - \varphi^0(\tau)|| \, d\tau$. Now Gronwall’s inequality implies the assertion. ■

Next we present the theory for general continuous linear skew-product flows and then specialize the results to perturbed linear systems.

Let $\Phi = (\theta, \varphi) : \mathbb{R} \times B \times \mathbb{R}^d \to B \times \mathbb{R}^d$ be a linear skew-product flow with continuous base flow $\theta : \mathbb{R} \times B \to B$. Throughout this subsection, $B$ is compact and $\theta$ is chain recurrent on $B$. Denote by $B \times \mathbb{R}^{d-1}$ the projective bundle and recall that $\Phi$ induces a dynamical system $\mathbb{P}\Phi : \mathbb{R} \times B \times \mathbb{P}^{d-1} \to B \times \mathbb{P}^{d-1}$. For $\varepsilon, T > 0$ an $(\varepsilon, T)$-chain $\zeta$ of $\mathbb{P}\Phi$ is given by $n \in \mathbb{N}, T_0, \ldots, T_n \geq T$, and $(u_0, p_0), \ldots, (u_n, p_n) \in B \times \mathbb{P}^{d-1}$ with $d(\mathbb{P}\Phi(T_i, u_i, p_i), (u_{i+1}, p_{i+1})) < \varepsilon$ for $i = 0, \ldots, n - 1$. For these systems we obtain an extension of a classical result of Conley that is due to Selgrade [34].

**Theorem 10.4** The projected flow $\mathbb{P}\Phi$ has a finite number of chain-recurrent components $M_1, \ldots, M_l$, $l \leq d$. These components form the finest Morse decomposition for $\mathbb{P}\Phi$, and they are linearly ordered $M_1 \prec \ldots \prec M_l$. Their lifts $\mathcal{V}_i := \mathbb{P}^{-1}M_i \subset B \times \mathbb{R}^d$ form a continuous bundle decomposition $B \times \mathbb{R}^d = \bigoplus_{i=1}^{l} \mathcal{V}_i$.  

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Before we sketch the proof of this theorem, we discuss how it relates to the previous results. The subbundle decomposition (also called a Whitney sum) shows that for every \( u \in B \) one has a decomposition into \( l \) linear subspaces \( \mathcal{V}^u_i := \{ x \in \mathbb{R}^d, (u, x) \in \mathcal{V}_i \} \),

\[
\mathbb{R}^d = \mathcal{V}^u_0 \oplus \ldots \oplus \mathcal{V}^u_l;
\]

the dimensions of the subspaces are independent of \( u \in B \), and \( \mathcal{V}^u_{i, t} = \{ \varphi(t, x, u), x \in \mathcal{V}^u_i \} \), \( t \in \mathbb{R} \). Hence it is a generalization of the decomposition (8.6) which is derived from a finest Morse decomposition in \( \mathbb{S}^1 \times \mathbb{P}^{d-1} \). In a moment we will discuss the relation to Lyapunov exponents.

For convenience, we write \( \mathcal{V} = B \times \mathbb{R}^d, \mathcal{P}V = B \times \mathbb{P}^{d-1} \) and identify the fibers \( \mathcal{V}^b := \{ b \} \times \mathbb{R}^d \) with \( \mathbb{R}^d \). For the proof of Theorem 10.4 we will need another characterization of Morse sets via attractors. For a flow on a compact metric space \( X \) a compact invariant set \( A \) is an attractor if it admits a neighborhood \( N \) such that \( \omega(N) = A \). We also allow the empty set as an attractor. For an attractor \( A \) the set \( A^* = \{ x \in X, \omega(x) \cap A = \emptyset \} \) is called the complementary repeller. Then for all \( x \notin A \cup A^* \) one has \( \omega(x) \subset A \) and \( \alpha(x) \subset A^* \).

The following result characterizes Morse decompositions via attractor-repeller sequences.

**Theorem 10.5** For a flow on a compact metric space \( X \) \( \log n \) collection of subsets \( \{ \mathcal{M}_1, ..., \mathcal{M}_n \} \) defines a Morse decomposition if and only if there is a strictly increasing sequence of attractors

\[
\emptyset = A_0 \subset A_1 \subset A_2 \subset ... \subset A_n = X;
\]

such that

\[
\mathcal{M}_{n-i} = A_{i+1} \cap A^*_i \text{ for } 0 \leq i \leq n-1.
\]

We proceed to analyze the relation to attractors.

**Definition 10.6** For \( Y \subset X \) define the chain limit set

\[
\Omega(Y) = \left\{ z \in X, \begin{array}{l} \text{there is } y \in Y \text{ such that for all } \varepsilon, T > 0 \text{ there is an } \varepsilon, T \text{-chain from } y \text{ to } z \end{array} \right\}.
\]

One easily sees that \( \omega(Y) \subset \Omega(Y) \).

**Proposition 10.7** For \( Y \subset X \) the set \( \Omega(Y) \) is the intersection of all attractors containing \( \omega(Y) \).

Next we begin the proof of Theorem 10.4 with the following lemma which establishes a relation between attractors in projective space and linearity.

**Lemma 10.8** Let \( A \) be an attractor in \( \mathcal{P}V \) and let \( v = (b, x), v' = (b, x') \in \mathcal{V} \) with \( x, x' \neq 0 \) be given and define the two-dimensional subspace \( L \) in \( \mathcal{V}^b \) by

\[
L = \{ cv + c'v', c, c' \in \mathbb{R} \}.
\]

(i) The set \( \mathcal{P}^{-1}A \) intersects each fiber \( \mathcal{V}^b \) in a linear space.

(ii) If \( \mathcal{P}v' \notin A \) and \( \mathcal{P}v \) is a boundary point of \( A \cap \mathcal{P}L \) relative to \( \mathcal{P}L \), then

\[
\lim_{t \to -\infty} \| \Phi(t, v) \| / \| \Phi(t, v') \| = 0. \tag{10.3}
\]

**Proof.** (ii) Choose \( \varepsilon \) with \( 0 < \varepsilon < d(A, A^*) \), where \( A^* := \{(b, p) \in \mathcal{P}V, \omega(b, p) \cap A = \emptyset \} \) is the complementary repeller of \( A \). By Lemma [14, Lemma B.1.17] there exists a \( \delta > 0 \) such that for all \( v_0 = (b, x_0), v_1 = (b, x_1) \in \mathcal{V} \) with \( x_0, x_1 \neq 0 \)

\[
\langle v_0, v_1 \rangle^2 / \| v_0 \|^2 \| v_1 \|^2 \geq 1 - \delta \text{ implies } d(\mathcal{P}v_0, \mathcal{P}v_1) \leq \varepsilon. \tag{10.4}
\]

Now suppose that (10.3) does not hold. Then there exist a sequence \( t_k \to -\infty \) and a constant \( K > 0 \) such that for all \( k \in \mathbb{N} \)

\[
\| \Phi(t_k, v') \| / \| \Phi(t_k, v) \| \leq K.
\]

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For $c \in \mathbb{R}$ with $|c|$ sufficiently small this implies that for all $k \in \mathbb{N}$

$$\frac{\langle \Phi(t_k, cv') + \Phi(t_k, v), \Phi(t_k, v) \rangle^2}{\| \Phi(t_k, cv') + \Phi(t_k, v) \|^2}$$

$$= \frac{c^2 \langle \Phi(t_k, v'), \Phi(t_k, v) \rangle^2 + 2c \| \Phi(t_k, v') \|^2 \langle \Phi(t_k, v'), \Phi(t_k, v) \rangle + \| \Phi(t_k, v) \|^4}{c^2 \| \Phi(t_k, v) \|^2}$$

$$\geq 1 - \delta.$$

Hence, by (10.4), for all $k \in \mathbb{N}$

$$d(\mathbb{P}\Phi(t_k, cv') + v, A) \leq \varepsilon$$

and thus $\omega^*(\mathbb{P}(cv' + v)) \not\subseteq A^*$. Therefore $\mathbb{P}(cv' + v) \in A$ for $|c|$ sufficiently small. This contradicts the assumption that $\mathbb{P}v$ is a boundary point of $A \cap \mathbb{P}L$ in $\mathbb{P}L$. Thus (10.3) holds.

(i) First we show that $A \cap \mathbb{P}L$ consists of a single point, if it contains a boundary point $\mathbb{P}v$ as earlier. For this purpose note that any point in $\mathbb{P}L \setminus \{\mathbb{P}v\}$ is given by $\mathbb{P}(v' + cv)$ for some $c \in \mathbb{R}$. It follows from (10.3) that

$$\lim_{t \to -\infty} \frac{\langle \Phi(t, v') + \Phi(t, cv), \Phi(t, v') \rangle^2}{\| \Phi(t, v') + \Phi(t, cv) \|^2}$$

$$= \lim_{t \to -\infty} \frac{\| \Phi(t, v') \|^2 + 2c \| \Phi(t, v') \|^2 \langle \Phi(t, v'), \Phi(t, v') \rangle + c^2 \langle \Phi(t, v), \Phi(t, v') \rangle^2}{c^2 \| \Phi(t, v) \|^2}$$

$$= 1.$$

Again by Lemma [14, Lemma B.1.17] this implies

$$\lim_{t \to -\infty} d(\mathbb{P}\Phi(t, v' + cv), \mathbb{P}\Phi(t, v')) = 0$$

and hence $\mathbb{P}(v' + cv) \notin A$. Therefore $A \cap \mathbb{P}L$ consists of a single point.

We have shown that for any two-dimensional subspace $L$ in $\mathcal{V}^b$ the set $A \cap \mathbb{P}L$ is empty, equals $\mathbb{P}L$, or consists of a single point. This implies that $\mathbb{P}^{-1}A$ intersects each fiber in a linear subspace.

**Lemma 10.9** If $L^b \subset \mathcal{V}^b$ is a linear subspace and $b' \in \Omega(b)$, then

$$L^{b'} = \left\{ v \in \mathcal{V}^{b'}, \ v = (b, x) \text{ with } x \neq 0 \implies \mathbb{P}v \in \Omega(\mathbb{P}L^b) \right\}$$

is a linear subspace of $\mathcal{V}^b$ and $\dim L^{b'} \geq \dim L^b$.

**Proof.** By Proposition 10.7, set $\Omega(\mathbb{P}L^b)$ is the intersection of attractors. Hence Lemma 10.8 implies that it intersects each fiber in a projective linear space. Therefore $L^{b'}$ is a linear subspace of $\mathcal{V}^b$. Now let us define the set $L^{b'}(\varepsilon, T)$ to be the closure of all points $v \in \mathcal{V}^{b'}$ such that there exists an $(\varepsilon, T)$-chain from some point in $\mathbb{P}L^b$ to $\mathbb{P}(-T, v)$. Thus for every $v \in L^{b'}(\varepsilon, T)$ there exists an $(\varepsilon, T)$-chain from some point in $\mathbb{P}L^b$ to $\mathbb{P}v$. This implies

$$\bigcap_{n \in \mathbb{N}} L^{b'}(1/n, n) \subset L^{b'}.$$
We conclude that the set \( \mathcal{L}(\varepsilon, T) \) of \( m \)-dimensional subspaces of \( \mathcal{V}^b \) contained in \( \mathcal{L}^b(\varepsilon, T) \) is nonempty for \( m = \dim L^b \). Therefore the intersection of the decreasing sequence \( \mathcal{L}(1/n, n) \) of nonempty compact sets is nonempty. This proves the statement of the lemma. ■

These lemmas imply that attractors in \( \mathbb{P} \mathcal{V} \) generate subbundles in \( \mathcal{V} \) if the flow on the base space is chain transitive.

**Proposition 10.10** Let \( A \) be an attractor in a vector bundle \( \mathbb{P} \mathcal{V} \) with chain transitive base space. Then

\[
\mathbb{P}^{-1}A = \{ v \in \mathcal{V}, v = (b, x) \text{ with } x \neq 0 \text{ implies } \mathbb{P}v \in A \}
\]

is a subbundle of \( \mathcal{V} \).

**Proof.** By [14, Lemma B.1.13] it suffices to show that \( \mathbb{P}^{-1}A \) is a closed subset of \( \mathcal{V} \) that intersects a fiber in a linear subspace \( A^b \), \( b \in B \), and \( \dim A^b \) is constant for \( b \in B \). Closedness is clear, by definition of \( A \). Furthermore, by Lemma 10.8, it follows that

\[
A^b = \{ v \in \mathcal{V}^b, v = (b, x) \text{ with } x \neq 0 \text{ implies } \mathbb{P}v \in A \}
\]

is a linear subspace of \( \mathcal{V}^b \). Because \( A \) is an attractor containing \( \omega(\mathbb{P}A^b) \), it follows by Proposition 10.7 that \( \Omega(\mathbb{P}A^b) \subset A \) and hence
\[
\{ v \in \mathcal{V}^b, v = (b, x) \text{ with } x \neq 0 \text{ implies } \mathbb{P}v \in \Omega(\mathbb{P}A^b) \} \subset A^b.
\]

Therefore we obtain from Lemma 10.9 that \( \dim A^b \geq \dim A^b \). Chain transitivity of \( B \) implies that \( \Omega(b) = B \) and hence \( \dim A^b \) is constant for \( b \in B \). ■

This result allows us to characterize the chain recurrent components of a projective linear flow over a chain recurrent base space.

**Proof of Theorem 10.4.** Note first that there is always a Morse decomposition of \( \mathbb{P} \Phi \): Define \( A_0 = \emptyset \), \( A_1 = \mathbb{P} \mathcal{V} \), and \( \mathcal{M}_1 = A_1 \cap A_0^c \); then a Morse decomposition is given by \( \{ \mathcal{M}_1 \} \). Next we claim that for every Morse decomposition \( \{ \mathcal{M}_1, ..., \mathcal{M}_n \} \) corresponding to an attractor sequence \( \emptyset = A_0 \subset A_1 \subset ... \subset A_n = \mathbb{P} \mathcal{V} \) the sets \( \mathbb{P}^{-1}\mathcal{M}_{n-i} = \mathbb{P}^{-1}A_{i+1} \cap \mathbb{P}^{-1}A_i^* \), \( i = 0, ..., n - 1 \), define a Whitney decomposition of \( \mathcal{V} \) into subbundles. For \( n = 1 \), this is obviously true. So we assume that the assertion is true for all vector bundles and all attractor sequences of length \( n - 1 \) and prove it for \( n \). Because \( \mathcal{M}_n = A_1 \) it is an attractor, and by Proposition 10.10 \( \mathbb{P}^{-1}A_1 \) and \( \mathbb{P}^{-1}A_1^* \) are subbundles. It is easily seen that \( \{ \mathcal{M}_1, ..., \mathcal{M}_{n-1} \} \subset A_1^* \) is a Morse decomposition of \( A_1^* \). Hence by the induction assumption \( \mathbb{P}^{-1}A_j, \ j = 2, ..., n \) form a Whitney decomposition of \( \mathbb{P}^{-1}A_1^* \). It remains to show that \( \mathbb{P}^{-1}A_1 \) and \( \mathbb{P}^{-1}A_1^* \) form a Whitney decomposition of \( \mathcal{V} \). Write \( A = A_1 \), choose \( b \in B \), and assume that the corresponding fibers of \( \mathbb{P}^{-1}A \) and \( \mathbb{P}^{-1}A^* \) have dimensions \( r \) and \( s \), respectively. Because \( A \) and \( A^* \) are disjoint, it suffices to prove that \( r + s \geq \dim \mathcal{V} = d \).

Fix \( b \in B \) and consider a subspace \( F_b^{d-s} \) of \( \mathcal{V} \) complementary to \( A_b^* \). By the definition of \( A^* \), one has \( \omega(\mathbb{P}F_b^{d-s}) \subset A \). Because \( A \) is an attractor, \( \Omega(\mathbb{P}F_b^{d-s}) \subset A \). Now chain transitivity in \( B \) and Lemma 10.9 show that for each \( b' \in B \), the set \( \Omega(\mathbb{P}F_b^{d-s}) \) meets \( \mathbb{P} \mathcal{V} \) in a subspace of dimension at least \( d - s \). Therefore \( r \geq d - s \) as claimed, and we obtain that

\[
\mathbb{P} \mathcal{V} = \mathbb{P}^{-1}A_1^* \oplus \mathbb{P}^{-1}A_1 = \mathbb{P}^{-1}\mathcal{M}_1 \oplus \mathbb{P}^{-1} \mathcal{M}_2 \oplus ... \oplus \mathbb{P}^{-1} \mathcal{M}_n.
\]

In order to see that there exists a finest Morse decomposition let \( \{ \mathcal{M}_1, ..., \mathcal{M}_n \} \) and \( \{ \mathcal{M}_1', ..., \mathcal{M}_m' \} \) be two Morse decompositions corresponding to the attractor sequences \( \emptyset = A_0 \subset A_1 \subset ... \subset A_n = \mathbb{P} \mathcal{V} \) and \( \emptyset = A_0' \subset A_1' \subset ... \subset A_m' = \mathbb{P} \mathcal{V} \). By Proposition 10.10, all \( \mathbb{P}^{-1}A_i, \mathbb{P}^{-1}A_i^* \) are subbundles of \( \mathcal{V} \); hence by a dimension argument, \( n, m \leq d \). By the result proven earlier it follows that \( \mathbb{P}^{-1}A_{n-i} = \mathbb{P}^{-1}A_{i+1} \cap \mathbb{P}^{-1}A_i^* \) and \( \mathbb{P}^{-1}A_{n-j} = \mathbb{P}^{-1}A_{j+1} \cap \mathbb{P}^{-1}A_j^* \) are subbundles. Their intersections \( \mathcal{M}_{ij} = \mathcal{M}_i \cap \mathcal{M}_j' \) define a Morse decomposition. This shows that a refinement of Morse decompositions of \( (\mathbb{P} \Phi) \) leads to finer Whitney decompositions of \( \mathcal{V} \). Therefore, again by a dimension argument, there exists a finest Morse decomposition \( \{ \mathcal{M}_1, ..., \mathcal{M}_l \} \) with \( 1 \leq l \leq \dim \mathcal{V} \).

As before, we introduce the **Lyapunov exponents** of a linear skew-product flow as \( \lambda(u, x) = \limsup_{t \to -\infty} \frac{1}{t} \ln ||\varphi(t, u, x)|| \), where \( \varphi(\cdot, u, x) \) denotes again a trajectory of the flow \( \Phi \). Unfortunately,
the Lyapunov spectrum of a linear skew-product flow $\Phi$ is rather difficult to handle, compare the
discussion in [14]. Instead of looking at exponential growth rates defined via trajectories, as we did
in the previous two sections, one considers exponential growth rates defined via chains.

**Definition 10.11** We define the *finite time exponential growth rate* of such a chain $\zeta$ (or
chain exponent) by

$$\lambda(\zeta) = \left( \sum_{i=0}^{n-1} T_i \right)^{-1} \sum_{i=0}^{n-1} (\ln \| \varphi(T_i, u, x_i) \| - \ln \| x_i \|),$$

where $x_i \in \mathbb{P}^{-1}(p_i)$.

Let $M \subset B \times \mathbb{P}^{d-1}$ be a chain recurrent component of the flow $\mathbb{P}\Phi$. Define the *Morse spectrum* of the flow
over $M$ as

$$\Sigma_M(M) = \left\{ \lambda \in \mathbb{R}, \text{ there exist sequences } \varepsilon_n \to 0, T_n \to \infty \text{ and } (\varepsilon_n, T_n)-chains \zeta_n \text{ in } M \text{ such that } \lim \lambda(\zeta_n) = \lambda \right\}$$

and the *Morse spectrum of the flow* as

$$\Sigma_M(\Phi) = \left\{ \lambda \in \mathbb{R}, \text{ there exist sequences } \varepsilon_n \to 0, T_n \to \infty \text{ and } (\varepsilon_n, T_n)-chains \zeta_n \text{ in the chain recurrent set of } \mathbb{P}\Phi \text{ such that } \lim \lambda(\zeta_n) = \lambda \right\}.$$

Define the *Lyapunov spectrum over $M$* as

$$\Sigma_L(M) = \{ \lambda(u, x), (u, x) \in M, x \neq 0 \}$$

and the *Lyapunov spectrum of the flow $\Phi$* as

$$\Sigma_L(\Phi) = \{ \lambda(u, x), (u, x) \in B \times \mathbb{R}^d, x \neq 0 \}.$$

The Morse spectrum has the following nice properties:

**Theorem 10.12** Let $\Phi : \mathbb{R} \times B \times \mathbb{R}^d \to B \times \mathbb{R}^d$ be a linear skew-product flow with continuous base
flow. The we have

(i) The Lyapunov spectrum and the Morse spectrum are defined on the Morse sets, i.e.
$\Sigma_L(\Phi) = \bigcup_{i=1}^{l} \Sigma_L(M_i)$ and $\Sigma_M(\Phi) = \bigcup_{i=1}^{l} \Sigma_M(M_i)$.

(ii) For each Morse set $M_i$, the Lyapunov spectrum is contained in the Morse spectrum, i.e.
$\Sigma_L(M_i) \subset \Sigma_M(M_i)$ for $i = 1, \ldots, l$.

(iii) For each Morse set its Morse spectrum is a closed, bounded interval $\Sigma_M(M_i) = [\kappa_i^*, \kappa_i]$, and $\kappa_i^*, \kappa_i \in \Sigma_L(M)$ for $i = 1, \ldots, l$.

(iv) The intervals of the Morse spectrum are ordered according to the order of the Morse sets, i.e. $M_i \subset M_j$ is equivalent to $\kappa_i^* < \kappa_j^*$ and $\kappa_i < \kappa_j$.

As an application of these results, we consider robust linear systems of the form $\Phi : \mathbb{R} \times B \times \mathbb{R}^d \to B \times \mathbb{R}^d$, given by a perturbed linear differential equation $\dot{x} = A(u(t))x := A_0x + \sum_{i=1}^{m} u_i(t)A_ix$, with $A_0, \ldots, A_m \in gl(d, \mathbb{R})$, $u \in \mathcal{U} = \{ u : \mathbb{R} \to U, \text{ integrable on every bounded interval} \}$ and $U \subset \mathbb{R}^m$ is compact, convex with $0 \in intU$. Explicit equations for the induced perturbed system on the projective space $\mathbb{P}^{d-1}$ can be obtained as follows: Let $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ be the unit sphere embedded into $\mathbb{R}^d$. The projected system on $\mathbb{S}^{d-1}$ is given by

$$\dot{s}(t) = h(u(t), s(t)), \ u \in \mathcal{U}, \ s \in \mathbb{S}^{d-1},$$

where

$$h(u, s) = h_0(s) + \sum_{i=1}^{m} u_i h_i(s) \text{ with } h_i(s) = (A_i - s^TA_is \cdot I)s, \ i = 0, 1, \ldots, m.$$
equation also describes the projected system on \( \mathbb{P}^{d-1} \). For the Lyapunov exponents one obtains in the same way

\[
\lambda(u, x) = \limsup_{t \to \infty} \frac{1}{t} \ln \| \phi(t, u, x) \| = \limsup_{t \to \infty} \frac{1}{t} \int_0^t q(u(\tau), s(\tau)) \, d\tau
\]

with

\[
q(u, s) = q_0(s) + \sum_{i=1}^m u_i q_i(s) \quad \text{with} \quad q_i(s) = (A_i - s^T A_i s \cdot I) \, s, \quad i = 0, 1, \ldots, m.
\]

For a constant perturbation \( u(t) \equiv u \in \mathbb{R} \) for all \( t \in \mathbb{R} \) the corresponding Lyapunov exponents \( \lambda(u, x) \) of the flow \( \Phi \) are the real parts of the eigenvalues of the matrix \( A(u) \) and the corresponding Lyapunov spaces are contained in the bundles \( \mathbb{P}^{-1} \mathcal{M}_i \). Similarly, if a perturbation \( u \in \mathcal{U} \) is periodic, the Floquet exponents of \( \dot{x} = A(u(t)) x \) are part of the Lyapunov (and hence of the Morse) spectrum of the flow \( \Phi \), and the Floquet spaces are contained in \( \mathbb{P}^{-1} \mathcal{M}_i \). (Note that the systems treated here can also be considered as ‘bilinear control systems’ and studied relative to their control behavior and (exponential) stabilizability - this is the point of view taken in \cite{[14]}.)

**Remark 10.13** For robust linear systems \( \dot{x} = A(u(t)) x \) as defined above the Morse spectrum is not much larger than the Lyapunov spectrum. Indeed, ‘generically’ the Lyapunov spectrum and the Morse spectrum agree, see \cite{[14]} for a precise definition of ‘generic’ in this context. This fact allows us to study stability and stabilizability of linear robust systems via the Morse spectrum.

In general, it is not possible to compute the Morse spectrum and the associated subbundle decompositions explicitly, even for relatively simple systems, and one has to revert to numerical algorithms, compare \cite{[14]}, Appendix D.

**Example 10.14** Let us consider, e.g., the linear oscillator with uncertain restoring force

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-1 & -2b
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + u(t)
\begin{bmatrix}
0 & 0 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

with \( u(t) \in [-\rho, \rho] \) and \( b > 0 \). Figure 1 shows the spectral intervals for this system depending on \( \rho \geq 0 \).

Of particular interest is the upper spectral interval \( \Sigma_{Mo}(\mathcal{M}_1) = [\kappa_1^*, \kappa_1] \), as its determines the robust stability of \( \dot{x} = A(u(t)) x \) (and stabilizability of the system if the set \( \mathcal{U} \) is interpreted as a set of admissible control functions, see \cite{[21]}).

**Definition 10.15** The stable, center, and unstable subbundles of \( \mathcal{U} \times \mathbb{R}^d \) associated with the perturbed linear system \( \dot{x} = A(u(t)) x \) are defined as

\[
L^- = \bigoplus \{ \mathbb{P}^{-1} \mathcal{M}_l, \kappa_j < 0 \}, \quad L^0 = \bigoplus \{ \mathbb{P}^{-1} \mathcal{M}_j, 0 \in [\kappa_j^*, \kappa_j] \}, \quad \text{and} \quad L^+ = \bigoplus \{ \mathbb{P}^{-1} \mathcal{M}_j, \kappa_j^* > 0 \},
\]

respectively.

**Theorem 10.16** The zero solution of \( \dot{x} = A(u(t)) x \) is exponentially stable for all perturbations \( u \in \mathcal{U} \) if and only if \( \kappa_1 < 0 \) if and only if \( L^- = \mathcal{U} \times \mathbb{R}^d \).

Comparing this theorem to Theorem 2.15 for constant matrices, Theorem 8.14 for periodic matrix functions and Theorem 9.10 for linear random systems, we see how parallel these theories are when one uses appropriate exponential growth rates and associated subspace decompositions.

More information can be obtained for robust linear systems if one considers its spectrum depending on a varying perturbation range: We introduce the family of varying perturbation ranges as \( U^\rho = \rho \mathcal{U} \) for \( \rho \geq 0 \). The resulting family of systems is

\[
\dot{x}^\rho = A(u^\rho(t)) x^\rho := A_0 x^\rho + \sum_{i=1}^m u_i^\rho(t) A_i x^\rho,
\]

with \( u^\rho \in \mathcal{U}^\rho = \{ u : \mathbb{R} \to U^\rho, \text{integrable on every bounded interval} \} \). As it turns out, the corresponding maximal spectral value \( \kappa_1(\rho) \) is continuous in \( \rho \). Hence we can define the (asymptotic)
stability radius of this family as 

\[ r = \inf \{ \rho \geq 0 \mid \text{there exists } u_0 \in \mathcal{U}^\rho \text{ such that } \dot{x}^\rho = A(u_0(t))x^\rho \text{ is not exponentially stable} \} \].

This stability radius is based on asymptotic stability under all time varying perturbations. Similarly one can introduce stability radii based on time invariant perturbations (with values in \( \mathbb{R}^m \) or \( \mathbb{C}^m \)) or on quadratic Lyapunov functions, compare [14], Chapter 11 and [25].

The stability radius plays a key role in the design of engineering systems if one is interested in guaranteed stability for all perturbations of a given size \( U \). We present two simple systems to illustrate this concept.

**Example 10.17** Linear oscillator with uncertain damping: Consider the oscillator

\[
\ddot{y} + 2(b + u(t))\dot{y} + (1 + c)y = 0
\]

with \( u(t) \in [-\rho, \rho] \) and \( c \in \mathbb{R} \). In equivalent first order form the system reads

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-1 - c & -2b
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + u(t) \begin{bmatrix}
0 & 0 \\
0 & -2
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}.
\]

Clearly, the system is not exponentially stable for \( c \leq -1 \) with \( \rho = 0 \), and for \( c > -1 \) with \( \rho \geq b \). It turns out that the stability radius for this system is

\[
r(c) = \begin{cases}
0 & \text{for } c \leq -1 \\
b & \text{for } c > -1.
\end{cases}
\]

**Example 10.18** Linear oscillator with uncertain restoring force: Here we look again at a system of the form

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-1 & -2b
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + u(t) \begin{bmatrix}
0 & 0 \\
0 & -1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

with \( u(t) \in [-\rho, \rho] \) and \( b > 0 \). (For \( b \leq 0 \) the system is unstable even for constant perturbations.) A closed form expression of the stability radius for this system is not available and one has to use
numerical methods for the computation of (maximal) Lyapunov exponents (or maxima of the Morse spectrum), compare [14], Appendix D. Figure 2 shows the (asymptotic) stability radius $r$, the stability radius under constant real perturbations $r_R$, and the stability radius based on quadratic Lyapunov functions $r_{Lf}$, all in dependence on $b > 0$, see [14], Example 11.1.12.

Figure 2: Stability radii of the linear oscillator with uncertain restoring force
Part III

Linearization

11 Linearization of Differential Equations

The local behavior of the dynamical system induced by a nonlinear differential equation can be studied via the linearization of the flow. At a fixed point of the nonlinear system the linearization is just a linear differential equation as studied in Sections 2 - 5. If the linearized system is hyperbolic, then the theorem of Grobman and Hartman, compare [19] and [24], states that the nonlinear flow is topologically conjugate to the linear flow. The invariant manifold theorem deals with those solutions of the nonlinear equation that are asymptotically attracted to (or repelled from) a fixed point. Basically these solutions live on manifolds that are described by nonlinear changes of coordinates of the linear stable (and unstable) subspaces.

Theorem 11.7 below describes the simplest form of the invariant manifold theorem at a fixed point. It can be extended to include a ‘center manifold’ (corresponding to the Lyapunov space with exponent 0). Furthermore, (local) invariant manifolds can be defined not just for the stable and unstable subspace, but for all Lyapunov spaces, see [10], [14], and [33] for the necessary techniques and precise statements.

Both, the Grobman-Hartman theorem as well as the invariant manifold theorem can be extended to time varying systems, i.e., to linear skew-product flows as described in Sections 7 - 10. The general situation is discussed in [10], the case of linearization at a periodic solution is covered in [33], random dynamical systems are treated in [3], and robust systems (control systems) are the topic of [14].

In this section we only consider linearization of a nonlinear (time-homogeneous) differential equation at a fixed point, using the theory presented in Sections 2 and 3. This should give the reader a taste of the results that can be expected. For details we refer to [5], [33], [10], [3], and [14].

We start by defining the basic objects discussed in this section.

A (nonlinear) differential equation in $\mathbb{R}^d$ is of the form $\dot{y} = f(y)$, where $f$ is a vector field on $\mathbb{R}^d$. We assume that $f$ is at least of class $C^1$ and that for all $y_0 \in \mathbb{R}^d$ the solutions $y(t,y_0)$ of the initial value problem $y(0,y_0) = y_0$ exist for all $t \in \mathbb{R}$. A point $p \in \mathbb{R}^d$ is a fixed point of the differential equation $\dot{y} = f(y)$ if $y(t,p) = p$ for all $t \in \mathbb{R}$.

The linearization of the equation $\dot{y} = f(y)$ at a fixed point $p \in \mathbb{R}^d$ is given by $\dot{x} = D_y f(p)x$, where $D_y f(p)$ is the Jacobian (matrix of partial derivatives) of $f$ at the point $p$.

A fixed point $p \in \mathbb{R}^d$ of the differential equation $\dot{y} = f(y)$ is called hyperbolic if $D_y f(p)$ has no eigenvalues on the imaginary axis, i.e. if the matrix $D_y f(p)$ is hyperbolic.

**Definition 11.1** Given a differential equation $\dot{y} = f(y)$ in $\mathbb{R}^d$ with flow $\Phi : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, hyperbolic fixed point $p$ and neighborhood $U(p)$. In this situation the **local stable manifold** and the **local unstable manifold** are defined as

$$W^s_{\text{loc}}(p) = \{q \in U : \lim_{t \to -\infty} \Phi(t,q) = p\} \quad \text{and} \quad W^u_{\text{loc}}(p) = \{q \in U : \lim_{t \to -\infty} \Phi(t,q) = p\},$$

respectively.

Note that the local stable (and unstable) manifolds can be extended to **global invariant manifolds** by following the trajectories, i.e.

$$W^s(p) = \bigcup_{t \geq 0} \Phi(-t,W^s_{\text{loc}}(p)) \quad \text{and} \quad W^u(p) = \bigcup_{t \geq 0} \Phi(t,W^u_{\text{loc}}(p)).$$

The following example shows that a nonlinear differential equation can have both, hyperbolic and non-hyperbolic fixed points.

**Example 11.2** Consider the nonlinear differential equation in $\mathbb{R}$ given by $\ddot{z} + z - z^3 = 0$, or in first order form in $\mathbb{R}^2$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ -y_1 + y_1^3 \end{bmatrix} = f(y).$$
The fixed points of this system are \( p_1 = [0, 0]^T \), \( p_2 = [1, 0]^T \), \( p_3 = [-1, 0]^T \). Computation of the linearization yields
\[
D_yf = \begin{bmatrix}
0 & 1 \\
-1 + 3y_1^2 & 0
\end{bmatrix}.
\]

Hence the fixed point \( p_1 \) is not hyperbolic, while \( p_2 \) and \( p_3 \) have this property.

The key result for the analysis of nonlinear flows in the neighborhood of a fixed point is the Grobman-Hartman Theorem. It says that locally nonlinear flows ‘look like’ their linearized flows:

**Theorem 11.3** (Grobman-Hartman) Consider a differential equation \( \dot{y} = f(y) \) in \( \mathbb{R}^d \) with flow \( \Phi : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d \). Assume that the equation has a hyperbolic fixed point \( p \) and denote the flow of the linearized equation \( \dot{x} = D_yf(p)x \) by \( \Psi : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d \). Then there exist neighborhoods \( U(p) \) of \( p \) and \( V(0) \) of the origin in \( \mathbb{R}^d \), and a homeomorphism \( h : U(p) \rightarrow V(0) \) such that the flows \( \Phi \mid_{U(p)} \) and \( \Psi \mid_{V(0)} \) are (locally) \( C^0 \)-conjugate, i.e., \( h(\Phi(t,y)) = \Psi(t,h(y)) \) for all \( y \in U(p) \) and \( t \in \mathbb{R} \) as long as the solutions stay within the respective neighborhoods.

Combining this theorem with Theorem 3.15 we obtain the typical behavior of a nonlinear flow around a hyperbolic fixed point as follows:

**Corollary 11.4** Consider a differential equation \( \dot{y} = f(y) \) in \( \mathbb{R}^d \) with flow \( \Phi : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d \). Assume that the equation has a hyperbolic fixed point \( p \). Then there exists a neighborhood \( U(p) \) on which \( \Phi \) is \( C^0 \)-equivalent to the flow of a linear differential equation of the type
\[
\dot{x}_s = -x_s, \quad x_s \in \mathbb{R}^{d_s},
\]
\[
\dot{x}_u = x_u, \quad x_u \in \mathbb{R}^{d_u},
\]
where \( d_s \) and \( d_u \) are the dimensions of the stable and the unstable subspace of \( D_yf(p) \), respectively, with \( d_s + d_u = d \).

Theorem 11.3 allows us to compare the flows of two nonlinear differential equations:

**Theorem 11.5** Given two differential equations \( \dot{y} = f_i(y) \) in \( \mathbb{R}^d \) with flows \( \Phi_i : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) for \( i = 1, 2 \). Assume that each \( \Phi_i \) has a hyperbolic fixed point \( p_i \).

(i) If the flows \( \Phi_1 \) and \( \Phi_2 \) are \( C^k \)-conjugate for some \( k \geq 1 \) in neighborhoods of the \( p_i \), then \( \sigma(D_yf_1(p_1)) = \sigma(D_yf_2(p_2)) \), i.e. the eigenvalues of the linearizations agree, compare Theorem 3.10 for the linear situation.

(ii) If the number of negative (or positive) Lyapunov exponents of \( D_yf_i(p_i) \) agrees, then the flows \( \Phi_i \) are locally \( C^0 \)-conjugate around the fixed points.

As an example we study a nonlinear system with two conjugacy classes of local flows.

**Example 11.6** Consider the nonlinear differential equation in \( \mathbb{R} \) given by \( \ddot{z} + \sin(z) + \dot{z} = 0 \), or in first order form in \( \mathbb{R}^2 \)
\[
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} = \begin{bmatrix}
y_2 \\
-\sin(y_1) - y_2
\end{bmatrix} = f(y).
\]

The fixed points of the system are \( p_n = [n\pi, 0]^T \) for \( n \in \mathbb{Z} \). Computation of the linearization yields
\[
D_yf = \begin{bmatrix}
0 & 1 \\
-\cos(y_1) & 0
\end{bmatrix}.
\]

Hence for the fixed points \( p_n \) with \( n \) even the eigenvalues are \( \mu_1, \mu_2 = -\frac{1}{2} \pm i\sqrt{\frac{3}{4}} \) with negative real part (or Lyapunov exponent), while at the fixed points \( p_n \) with \( n \) odd one obtains as eigenvalues \( \nu_1, \nu_2 = -\frac{1}{2} \pm \sqrt{\frac{3}{4}} \), resulting in one positive and one negative eigenvalue. Hence the flow of the differential equation is locally \( C^0 \)-conjugate around all fixed points with even \( n \), and around all fixed points with odd \( n \), while the flows around, e.g., \( p_0 \) and \( p_1 \) are not conjugate.
Our discussion so far has concentrated on the local behavior of nonlinear systems around hyperbolic fixed points. Corollary 11.4 shows how the dimensions of the stable and unstable subspaces of the linearization enter into the picture. But what happens to these subspaces from the nonlinear point-of-view? The answer is given by the invariant manifold theorem.

**Theorem 11.7** (Invariant Manifold Theorem) Consider a differential equation \( \dot{y} = f(y) \) in \( \mathbb{R}^d \) with flow \( \Phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \). Assume that the equation has a hyperbolic fixed point \( p \) and denote the linearized equation by \( \dot{x} = D_y f(p) x \).

(i) There exists a neighborhood \( U(p) \) in which the flow \( \Phi \) has a local stable manifold \( W_{\text{loc}}^s(p) \) and a local unstable manifold \( W_{\text{loc}}^u(p) \).

(ii) Denote by \( L^- \) (and \( L^+ \)) the stable (and unstable, respectively) subspace of \( D_y f(p) \), compare the definitions in Subsection 1. The dimensions of \( L^- \) (as a linear subspace of \( \mathbb{R}^d \)) and of \( W_{\text{loc}}^s(p) \) (as a topological manifold) agree, similarly for \( L^+ \) and \( W_{\text{loc}}^u(p) \).

(iii) The stable manifold \( W_{\text{loc}}^s(p) \) is tangent to the stable subspace \( L^- \) at the fixed point \( p \), similarly for \( W_{\text{loc}}^u(p) \) and \( L^+ \).

This theorem says that the invariant manifolds are 'just' nonlinear transformations of the stable and unstable subspaces. For applications of this circle of ideas we refer the reader again to [5] and [33] for nonlinear differential equations as discussed in this section, to [10] for a general result in the context of skew-product flows, to [3] for random systems, and to [14] for robust and control systems.

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**References**


