M201  OBJECTIVES FOR FINAL EXAM

This exam will cover Chapter 1, 2 and 3 of Basic Analysis. The new topics to review include the following.— This includes and incorporates all covered in Midterms.

1. STATE THE DEFINITIONS AND USE THEM IN PROOFS

(1) Real Numbers
- Extended Real Numbers, symbols $\infty, -\infty$
- Intervals in $\mathbb{R}$: open, closed and half-open intervals, bounded and unbounded intervals
- Maximum and minimum elements of a set
- The absolute value function
- Bounded real-valued function.
- Supremum and infimum of a bounded function

(2) Sequences
- Sequence (of real numbers)
- Bounded sequence
- Convergence of a sequence
- Limit of a sequence
- Convergent sequence, divergent sequence
- Monotone increasing sequence; monotone decreasing sequence; monotone sequence
- Tail of a sequence
- Subsequence of a sequence
- Limit superior and limit inferior of a sequence
- Cluster point of a set $S \subseteq \mathbb{R}$
- Cauchy sequence

(3) Series:
- Series, partial sum;
- Convergence and divergence of series,
- Absolutely convergent series, conditionally convergent series.

(4) Limits of Functions, Continuity
- Cluster point of a set $S \subseteq \mathbb{R}$.
- Limit of a function at a cluster point of its domain.
- One-sided limit of a function at a point.
- Continuity of a function at a point of its domain.
- Continuous function on a set.

2. STATE THE THEOREMS AND USE THEM IN PROOFS

(1) Real Numbers
- Archimedean property
- Corollary: $\inf\{\frac{1}{n} : n \in \mathbb{N}\} = 0$.
- For $A \subseteq \mathbb{R}$, behavior of $\sup A$ and $\inf A$ under translation and dilation.
- Approximation property of $\sup S$ (and $\inf S$).
- Properties of absolute value
- The triangle inequality and the reverse triangle inequality

(2) Sequences
- Uniqueness of the limit of a convergent sequence
- Boundedness of convergent sequences
- A monotone sequence converges if and only if it is bounded.
- A bounded monotone increasing sequence converges to its supremum.
- A bounded monotone decreasing sequence converges to its infimum.
- A sequence converges if and only if any tail sequence converges;
- A convergent sequence and any of its tails have the same limit.
- Any subsequence of a convergent sequence is convergent and has the same limit as the original sequence.
- The squeeze lemma
- Limits and algebraic operations
- Existence (or nonexistence) of limits for sequences $c^n$
- Ratio test for sequences
- A bounded sequence has subsequences converging to its limit superior and to its limit inferior.
- A bounded sequence converges if and only if its limit superior is equal to its limit inferior.
- Bolzano-Weierstrass Theorem (two versions)
- A Cauchy sequence is bounded
- A sequence is convergent if and only if it satisfies Cauchy’s criterion.

(3) Series
- A series converges if and only if every one of its tail series converges.
- Cauchy convergence criterion for series.
- Necessary condition for convergence of a series (also known as the divergence test).
- Convergence criterion (and sum when convergent) for geometric series.
- Divergence of the harmonic series.
- Linear operations on convergent series.
- An absolutely convergent series is convergent.
- Comparison test.
- Necessary and sufficient condition for convergence of the series $\zeta(p) = \sum_{n=1}^{\infty} n^{-p}$.
- Ratio test for series.
- Triangle inequality for series.
- Limit comparison test.

(4) Limits of Functions, Continuity
- Necessary and sufficient condition for a point to be a cluster point of a set.
- Uniqueness of the limit of a function at a point.
- A function has a limit at a point if and only if it has a sequential limit at that point.
- Squeeze lemma for limits of functions.
- Limits of sums, products and quotients.
- Continuity is equivalent to sequential continuity.
- The composition of continuous functions is continuous.
- Examples of extremely discontinuous functions.
- A continuous function $f: [a, b] \to \mathbb{R}$ is bounded.
- Maximum-minimum theorem for continuous functions.
- Bolzano’s intermediate value theorem for continuous functions.