M201 OBJECTIVES FOR MIDTERM EXAM 2

This midterm covers Chapters 7 through 13 in (BP). Of course, this will include and incorporate everything covered in Exam 1.

1. STATE THE DEFINITIONS AND USE THEM IN PROOFS

(1) Numbers
   - Rational number, irrational number, real numbers
   - Binomial coefficients

(2) Relations and Functions
   - Relation, function
   - Domain, codomain and range of a function;
   - Equality of functions
   - Injective function, surjective function, bijective function
   - Composition of functions
   - Identity function on a set
   - Inverse of a relation; inverse function for a bijection
   - Image of a set in the domain of a function;
   - Preimage (inverse image) of a subset of a function’s codomain

(3) Cardinality
   - Sets having the same cardinality
   - Sets having unequal cardinalities
   - Countably infinite set
   - Uncountable set
   - Meaning of $|A| = |B|$ and $|A| < |B|$ when $A, B$ are sets.

2. USE PROOF TECHNIQUES

(1) Prove biconditional statements.
(2) Prove existence and uniqueness statements.
(3) Prove set-membership statements.
(4) Prove that a set is a subset of, or is equal to, another set.
(5) To disprove a statement $P$:
    - Either prove $\sim P$; or
    - assume $P$ and derive a contradiction.
(6) Use a counterexample to disprove a universal statement.
(7) Construct proofs by induction.
(8) Prove that a function is injective, surjective, or bijective.
3. State the Theorems and Use Them in Proofs

(1) Integers
- Division algorithm for integers
- Arithmetical properties of congruence of integers
- For integers $a$ and $b$ not both zero, the $\text{gcd}(a, b)$ is the smallest positive integral linear combination $ma + nb$ of $a$ and $b$.
- Euclid’s Lemma: If $a$ and $b$ are integers and $p$ is a prime such that $p | ab$, then $p | a$ or $p | b$.
- Properties of binomial coefficients:
  - Pascal triangle relation: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.
  - Binomial coefficient $\binom{n}{k}$ is the number of $k$-element subsets of an $n$-element set; and so, the sum of the coefficients over all $k$ from 0 to $n$ constitutes the cardinality of the power set of an $n$-set.
- Factorial formula for binomial coefficients

(2) Functions
- Composition of functions is associative.
- The composition of injective functions is injective;
- The composition of surjective functions is surjective.
- A function $f$ is bijective if and only if the inverse relation $f^{-1}$ is a function.
- Inverse functions preserve all set operations while functions do not necessarily preserve set operations:
  - If $f : A \to B$ and $W, X \subseteq A$ and $Y \subseteq B$, then
    - $f(W \cap X) \subseteq f(W) \cap f(X)$, but equality need not hold.
    - $f(W \cup X) = f(W) \cup f(X)$.
    - $X \subseteq f^{-1}(f(X))$, but equality need not hold.
    - $Y \subseteq f(f^{-1}(Y))$, but equality need not hold.
  (cf. Theorem 12.4 in (BP).)

(3) Cardinality
- $|N| = |Z|$.
- $|R| = |(0, ?)| = |(0, 1)|$.
- The set of rational numbers $\mathbb{Q}$ is countably infinite.
- There exists no bijection $f : \mathbb{N} \to \mathbb{R}$. 