

Fully invariant and verbal congruence relations

CLIFFORD BERGMAN AND JOEL BERMAN

ABSTRACT. A congruence relation θ on an algebra \mathbf{A} is fully invariant if every endomorphism of \mathbf{A} preserves θ . A congruence θ is verbal if there exists a variety \mathcal{V} such that θ is the least congruence of \mathbf{A} such that $\mathbf{A}/\theta \in \mathcal{V}$. Every verbal congruence relation is known to be fully invariant. This paper investigates fully invariant congruence relations that are verbal, algebras whose fully invariant congruences are verbal, and varieties for which every fully invariant congruence in every algebra in the variety is verbal.

1. Introduction

One of the central tools of universal algebra, in fact, all of algebra, is the free object. Every variety contains free algebras. This fact is a component of countless arguments in the subject.

In the usual construction we start with an algebra of terms and take an appropriate quotient. The congruence inducing this quotient is the smallest one yielding a member of the desired variety.

But there is no reason to restrict our attention to term algebras. Applying this strategy to an arbitrary algebra \mathbf{A} and variety \mathcal{V} yields the “most general” homomorphic image of \mathbf{A} that lies in \mathcal{V} . This technique is very familiar to group-theorists. For example, the *abelianization* of \mathbf{A} , i.e., the largest Abelian quotient of \mathbf{A} , is an instance of this construction. In group theory, the normal subgroups that induce these quotients are collectively called *verbal*. We adopt that terminology in this paper.

Definition 1.1. Let \mathbf{A} be an algebra and \mathcal{V} a variety with the same similarity type as \mathbf{A} .

- $\Lambda_{\mathcal{V}}^{\mathbf{A}} := \{\theta \in \text{Con } \mathbf{A} : \mathbf{A}/\theta \in \mathcal{V}\}$.
- $\lambda_{\mathcal{V}}^{\mathbf{A}} := \bigcap \Lambda_{\mathcal{V}}^{\mathbf{A}}$.
- A congruence relation of the form $\lambda_{\mathcal{V}}^{\mathbf{A}}$ is called a *verbal congruence relation* of the algebra \mathbf{A} .
- $\text{Con}_{\text{ve}} \mathbf{A}$ denotes the set of all verbal congruences of \mathbf{A} .

Since \mathcal{V} is closed under subdirect products, $\mathbf{A}/\lambda_{\mathcal{V}}^{\mathbf{A}}$ lies in \mathcal{V} . Furthermore, by definition, $\lambda_{\mathcal{V}}^{\mathbf{A}}$ is the least congruence relation on \mathbf{A} with this property. Put another way, $\Lambda_{\mathcal{V}}^{\mathbf{A}}$ consists of the congruences in the interval sublattice $[\lambda_{\mathcal{V}}^{\mathbf{A}}, 1_{\mathbf{A}}]$

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of **Con** \mathbf{A} . In any algebra \mathbf{A} we have $1_A = \lambda_{\mathcal{T}}^{\mathbf{A}}$ and $0_A = \lambda_{\mathbf{V}(\mathbf{A})}^{\mathbf{A}}$, where \mathcal{T} is the trivial variety and $\mathbf{V}(\mathbf{A})$ is the variety generated by \mathbf{A} . If the algebra \mathbf{A} is understood from the context, then we suppress the superscripts and write $\Lambda_{\mathcal{V}}$ and $\lambda_{\mathcal{V}}$.

For arbitrary varieties \mathcal{V} and \mathcal{W} we have

$$\begin{aligned}\lambda_{\mathcal{V}} \vee \lambda_{\mathcal{W}} &= \lambda_{\mathcal{V} \wedge \mathcal{W}} \\ \lambda_{\mathcal{V}} \wedge \lambda_{\mathcal{W}} &\supseteq \lambda_{\mathcal{V} \vee \mathcal{W}}\end{aligned}\tag{1}$$

although equality in the second of these need not hold in general (see Example 2.12). Thus the set $\text{Con}_{\text{ve}} \mathbf{A}$ is the universe of a complete join subsemilattice of $\text{Con } \mathbf{A}$.

There is another way to approach the definition of verbal congruence relation. Let \mathbf{A} be an algebra and Σ a set of equations of the same similarity type as \mathbf{A} . Define

$$\lambda_{\Sigma}^{\mathbf{A}} := \text{Cg}^{\mathbf{A}} \left\{ (s(a_1, \dots, a_n), t(a_1, \dots, a_n)) : s \approx t \in \Sigma, a_1, \dots, a_n \in A \right\}.$$

It is easy to see that if $\mathcal{V} = \text{Mod}(\Sigma)$, then the congruence $\lambda_{\mathcal{V}}^{\mathbf{A}}$ of Definition 1.1 coincides with $\lambda_{\Sigma}^{\mathbf{A}}$. This usage explains the origin of the term ‘‘verbal’’.

Let us return to the abelianization example we mentioned earlier. Suppose \mathbf{G} is a group and \mathcal{A} is the variety of Abelian groups. The congruence $\lambda_{\mathcal{A}}^{\mathbf{G}}$ corresponds to a normal subgroup on \mathbf{G} . This normal subgroup is usually written $[G, G]$ and is called the derived subgroup.

As another example, consider the set Σ_n consisting of the group identities together with $x^n \approx e$, where e denotes the identity element. Then $\mathbf{G}/\lambda_{\Sigma_n}$ is the largest quotient of \mathbf{G} of exponent n . Since exponent 2 implies Abelian, we have $\lambda_{\mathcal{A}}^{\mathbf{G}} \leq \lambda_{\Sigma_2}^{\mathbf{G}}$.

Although the concept of verbal congruence is a natural one, there are aspects that are a bit awkward. For one thing, the property of being verbal is not completely internal to the algebra since it refers to a variety, which may not be closely related to the algebra. For another, as we noted earlier, the intersection of two verbal congruences may not be verbal. However, there is a second notion, closely related to verbality, that addresses these concerns.

Definition 1.2. Let \mathbf{A} be an algebra and $\text{End}(\mathbf{A})$ the set of endomorphisms of \mathbf{A} .

- A congruence relation θ on \mathbf{A} is called *fully invariant* if and only if every $e \in \text{End}(\mathbf{A})$ preserves θ , i.e., $(a, b) \in \theta \implies (e(a), e(b)) \in \theta$.
- $\text{Con}_{\text{fi}} \mathbf{A}$ denotes the set of all fully invariant congruences of \mathbf{A} .

Note that $\text{Con}_{\text{fi}} \mathbf{A}$ is the congruence lattice of the algebra \mathbf{A}^+ obtained from \mathbf{A} by adjoining every endomorphism of \mathbf{A} as a fundamental operation of \mathbf{A}^+ . Hence $\text{Con}_{\text{fi}} \mathbf{A} = \text{Con}(\mathbf{A}^+)$ is a complete sublattice of $\text{Con } \mathbf{A}$ and is an algebraic lattice.

The following proposition is well-known. A proof can be found in [1, Theorem 4.59].

Proposition 1.3. *Let \mathcal{V} be a variety and \mathbf{A} an algebra of the same similarity type. Then $\lambda_{\mathcal{V}}$ is a fully invariant congruence of \mathbf{A} .*

The converse of Proposition 1.3 is known to be false in general as Examples 4.1, 4.6, and 4.8 below show. The converse does hold for free algebras and this fact about free algebras is a central component in Birkhoff's theorem on the equality of varieties and equational classes. Proposition 1.4 follows from Theorem 3.4 below.

Proposition 1.4. *Let \mathcal{V} be a variety, \mathbf{F} a free algebra in \mathcal{V} , and θ a fully invariant congruence on \mathbf{F} . Then $\theta = \lambda_{\mathcal{W}}$ for some subvariety \mathcal{W} of \mathcal{V} .*

This paper is motivated by our attempt to understand the circumstances under which the converse of Proposition 1.3 holds. To that end, we make the following definition.

Definition 1.5. An algebra \mathbf{A} is called *verbose* if every fully invariant congruence relation on \mathbf{A} is verbal. A variety \mathcal{V} is *verbose* if every algebra in \mathcal{V} is verbose.

Section 2 contains some general facts about fully invariant congruences and verbal congruences. The focus is on necessary and sufficient conditions for a given congruence to be fully invariant or verbal. Section 3 deals with verbose algebras, and Section 4 with verbose varieties. Section 5 contains results concerning arithmetical semisimple varieties.

In general we follow the notational conventions of [1] or [15]. For a mapping f , we use $\vec{f}(X)$ to denote the direct image of the subset X under f , i.e., $\{f(x) : x \in X\}$. Similarly, for a binary relation θ , we write $\vec{f}(\theta) = \{(f(x), f(y)) : (x, y) \in \theta\}$. Thus, a congruence θ is fully invariant if $\vec{e}(\theta) \subseteq \theta$, for all endomorphisms e . For the inverse image of a subset Y under f we write $\overleftarrow{f}(Y) = \{x : f(x) \in Y\}$.

2. Fully invariant and verbal congruences

We begin with some tools that can be used to determine whether a congruence is fully invariant.

Lemma 2.1. *Let \mathbf{A} be an algebra with $e \in \text{End}(\mathbf{A})$ and $\theta \in \text{Con } \mathbf{A}$.*

- (1) *If $\ker e \geq \theta$, then $\vec{e}(\theta) \subseteq \theta$.*
- (2) *If e is one-to-one, then $\vec{e}(\theta)$ is a congruence relation of the subalgebra $\vec{e}(\mathbf{A})$.*
- (3) *If e is an automorphism of \mathbf{A} , then $\vec{e} : \text{Con } \mathbf{A} \rightarrow \text{Con } \mathbf{A}$ is a lattice isomorphism of $\text{Con } \mathbf{A}$.*
- (4) *Suppose $\vec{e}(\theta) \subseteq \theta$. Then $e_{\theta} : \mathbf{A}/\theta \rightarrow \mathbf{A}/\theta$, given by $(e_{\theta})(a/\theta) = e(a)/\theta$, is an endomorphism.*
- (5) *Suppose $\theta \leq \tau$ in $\text{Con } \mathbf{A}$. If θ is fully invariant on \mathbf{A} and τ/θ is fully invariant on \mathbf{A}/θ , then τ is fully invariant on \mathbf{A} .*

Proof. The proofs of these facts are straightforward. We only prove item (5). Let $e \in \text{End}(\mathbf{A})$ be arbitrary. Then e_θ is an endomorphism of \mathbf{A}/θ by Lemma 2.1(4). Thus $\vec{e}_\theta(\tau/\theta) \subseteq \tau/\theta$. If $(x, y) \in \tau$, then $(x/\theta, y/\theta) \in \tau/\theta$, and so $(e_\theta(x/\theta), e_\theta(y/\theta)) \in \tau/\theta$. Hence $(e(x)/\theta, e(y)/\theta) \in \tau/\theta$ and therefore $(e(x), e(y)) \in \tau$. \square

In general, the property of being fully invariant does not behave well with respect to the formation of subalgebra, products or homomorphic image. Property (5) above is the only positive assertion we can find.

On the other hand, the location of a congruence in the congruence lattice does sometimes imply full invariance. Obviously, both the smallest and largest congruences have this property. The next two theorems tell us a bit more in this regard.

Theorem 2.2. *The monolith of any finite subdirectly irreducible algebra is fully invariant.*

Proof. Let \mathbf{A} be a finite subdirectly irreducible algebra with monolith μ . Consider $e \in \text{End}(\mathbf{A})$. Suppose $\ker e \neq 0$. Then, since \mathbf{A} is subdirectly irreducible, $\ker e \geq \mu$, so $\vec{e}(\mu) \subseteq \mu$ by Lemma 2.1.

If $\ker e = 0_A < \mu$, then e is one-to-one and hence an automorphism since \mathbf{A} is finite. Because e is an automorphism, it follows that $\vec{e}(\mu)$ is a congruence relation of \mathbf{A} , and it is an atom in the congruence lattice of \mathbf{A} by Lemma 2.1. Hence $\vec{e}(\mu)$ must be μ , the unique atom of $\text{Con } \mathbf{A}$. So in all possible cases, $\vec{e}(\mu) \subseteq \mu$. \square

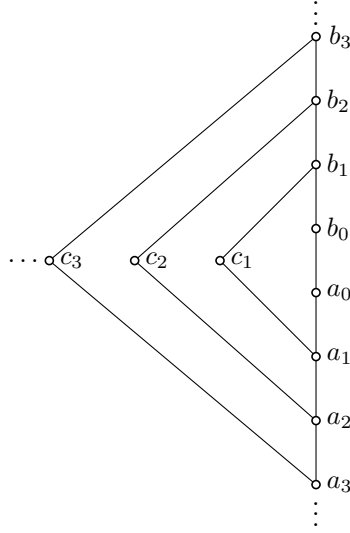
Theorem 2.3. *Let \mathbf{A} be a finite algebra. Suppose $\theta_0, \theta_1, \dots, \theta_m \in \text{Con } \mathbf{A}$ with $0_A = \theta_0 \prec \theta_1 \prec \dots \prec \theta_m$ and $\theta_m < \tau$ for all $\tau \in \text{Con } \mathbf{A} \setminus \{\theta_0, \theta_1, \dots, \theta_m\}$. Then each θ_i is fully invariant.*

Proof. The congruence $\theta_0 = 0_A$ is always fully invariant, and θ_1 is fully invariant by Theorem 2.2. To show that θ_2 is fully invariant, observe that θ_1 is fully invariant in \mathbf{A} and that θ_2/θ_1 is fully invariant in \mathbf{A}/θ_1 , since \mathbf{A}/θ_1 is subdirectly irreducible with monolith θ_2/θ_1 . Therefore θ_2 is fully invariant by Lemma 2.1(5). By iterating this argument each θ_i is shown to be fully invariant. \square

Corollary 2.4. *If \mathbf{A} is any finite algebra whose congruence lattice is a chain, then every congruence relation of \mathbf{A} is fully invariant.*

The finiteness hypothesis in Theorem 2.2 is necessary as the following example shows.

Example 2.5. Let \mathbf{N}_ω be the infinite lattice in Figure 1. This lattice is discussed in [14] and [3] where \mathbf{N}_ω is observed to be subdirectly irreducible with monolith μ , and μ has exactly one nontrivial congruence class $\{a_0, b_0\}$.

FIGURE 1. \mathbf{N}_ω

The lattice join operation on \mathbf{N}_ω is given by

$$a_i \vee a_k = a_{\min\{i,k\}}, \quad b_i \vee b_k = b_{\max\{i,k\}}, \quad c_i \vee c_k = b_{\max\{i,k\}},$$

$$a_i \vee b_k = b_k, \quad b_i \vee c_k = b_{\max\{i,k\}}, \quad a_i \vee c_k = \begin{cases} c_k, & \text{if } i \geq k, \\ b_k, & \text{if } i < k, \end{cases}$$

and lattice meet behaves in a dual manner. Suppose $e : N_\omega \rightarrow N_\omega$ is defined by $e(a_i) = a_{i+1}$, $e(b_i) = b_{i+1}$ and $e(c_i) = c_{i+1}$. It is easily checked that e is an endomorphism of \mathbf{N}_ω . For example,

$$e(a_i) \vee e(c_k) = a_{i+1} \vee c_{k+1} = \begin{cases} c_{k+1}, & \text{if } i+1 \geq k+1, \\ b_{k+1}, & \text{if } i+1 < k+1 \end{cases} = \begin{cases} e(c_k), & \text{if } i \geq k, \\ e(b_k), & \text{if } i < k \end{cases}$$

$$= e(a_i \vee c_k).$$

Then μ is not fully invariant since $(e(a_0), e(b_0)) \notin \mu$.

The next lemma provides a sufficient condition for a principal congruence relation to be fully invariant. It will be useful in showing that the monoliths of certain infinite subdirectly irreducible algebras are fully invariant.

Lemma 2.6. *Let \mathbf{A} be an algebra and $\theta = \text{Cg}^{\mathbf{A}}(a, b)$.*

- (1) *If $(e(a), e(b)) \in \theta$ for each $e \in \text{End}(\mathbf{A})$, then θ is fully invariant.*
- (2) *Let $0 \in A$ be the value of a constant term on \mathbf{A} . Suppose there is a unary term f and an element $a \in A$ such that $f(a) = 0$ and*

$$f(x) = 0 \implies (x, 0) \in \text{Cg}^{\mathbf{A}}(a, 0).$$

Then $\text{Cg}^{\mathbf{A}}(a, 0)$ is fully invariant.

Proof. For the first item observe that if there is a Malcev chain of unary polynomials witnessing $(x, y) \in \text{Cg}^{\mathbf{A}}(a, b)$, then there is a Malcev chain of unary polynomials witnessing $(e(x), e(y)) \in \text{Cg}^{\mathbf{A}}(e(a), e(b))$. So $(e(x), e(y)) \in \text{Cg}^{\mathbf{A}}(e(a), e(b)) \subseteq \text{Cg}^{\mathbf{A}}(a, b)$.

For the second item, write $\theta = \text{Cg}(a, 0)$. By (1) we must show that for any endomorphism e of \mathbf{A} , we have $(e(a), e(0)) \in \theta$. Since 0 is a constant, $e(0) = 0$. Since $f(a) = 0$ we get $e(f(a)) = e(0) = 0$. On the other hand, $f(e(a)) = e(f(a)) = 0$, so by assumption $e(a) \equiv 0 \pmod{\theta}$. \square

Is the monolith of a finite subdirectly irreducible algebra always verbal? The answer to this question is affirmative for the variety of groups, as shown by L. Kovács and M. Newman in [12, 13]. We extend the Kovács and Newman result to finite subdirectly irreducible algebras in congruence-modular varieties. Our proof adapts that of Kovács and Newman to the more general congruence-modular case by utilizing a result of E. Kiss [11].

A *section* of \mathbf{A} is an algebra \mathbf{B}/θ in which $\mathbf{B} \leq \mathbf{A}$ and $\theta \in \text{Con}(\mathbf{B})$. The section is *proper* unless $B = A$ and $\theta = 0$. We shall write $(\mathbf{HS})^*(\mathbf{A})$ for the set of proper sections of \mathbf{A} . The algebra \mathbf{A} is *critical* if A is finite and $\mathbf{A} \notin \mathbf{V}((\mathbf{HS})^*(\mathbf{A}))$.

A set, \mathcal{S} , of algebras is *closed under monolithic sections* if every subdirectly irreducible section of a member of \mathcal{S} is already in \mathcal{S} . The monolith of a subdirectly irreducible algebra \mathbf{A} is denoted $\mu_{\mathbf{A}}$. The class of all subdirectly irreducible members of a variety \mathcal{V} is denoted \mathcal{V}_{si} .

Lemma 2.7 (Kiss). *Let \mathcal{S} be a finite set of finite subdirectly irreducible algebras, closed under monolithic sections. Assume that $\mathcal{V} = \mathbf{V}(\mathcal{S})$ is congruence-modular. Then for every $\mathbf{A} \in \mathcal{V}_{\text{si}}$ we have*

$$\mathbf{A}/\mu_{\mathbf{A}} \in \mathbf{V}\{\mathbf{B}/\mu_{\mathbf{B}} : \mathbf{B} \in \mathcal{S}\}.$$

Theorem 2.8. *Let \mathbf{A} be a finite subdirectly irreducible algebra generating a congruence-modular variety. Then the monolith of \mathbf{A} is verbal.*

Proof. By the nature of the monolith, proving the theorem amounts to showing that $\mathbf{A} \notin \mathbf{V}(\mathbf{A}/\mu_{\mathbf{A}})$. If \mathbf{A} is critical, this holds *a fortiori*. So we can assume that \mathbf{A} is noncritical. Let \mathcal{T} be the set of proper monolithic sections of \mathbf{A} , i.e., the subdirectly irreducible members of $(\mathbf{HS})^*(\mathbf{A})$. Then $(\mathbf{HS})^*(\mathbf{A}) \subseteq \mathbf{V}(\mathcal{T})$. To see this, observe that a proper section of \mathbf{A} is of the form \mathbf{B}/θ . Write $\theta = \theta_1 \wedge \cdots \wedge \theta_n$, in which each θ_i is completely meet-irreducible. Then $\mathbf{B}/\theta_i \in \mathcal{T}$, for each $i \leq n$, so $\mathbf{B}/\theta \in \mathbf{V}(\mathcal{T})$. Thus, since \mathbf{A} is noncritical, $\mathbf{A} \in \mathbf{V}((\mathbf{HS})^*(\mathbf{A})) = \mathbf{V}(\mathcal{T})$.

Let \mathcal{S} be a minimal subset of \mathcal{T} , closed under monolithic sections, and such that $\mathbf{A} \in \mathbf{V}(\mathcal{S})$. Let \mathbf{D} be of maximal order in \mathcal{S} and set $\mathcal{S}_0 = \mathcal{S} - \{\mathbf{D}\}$. Then \mathcal{S}_0 is still closed under monolithic sections, so by the minimality of \mathcal{S} , $\mathbf{A} \notin \mathbf{V}(\mathcal{S}_0)$. By Lemma 2.7, $\mathbf{A}/\mu_{\mathbf{A}} \in \mathbf{V}\{\mathbf{B}/\mu_{\mathbf{B}} : \mathbf{B} \in \mathcal{S}\}$. Let $\mathbf{B} \in \mathcal{S}$. Write $\mu_{\mathbf{B}} = \theta_1 \wedge \cdots \wedge \theta_n$, in which each θ_i is completely meet-irreducible. By the

choice of \mathbf{D} , $|D| \geq |B|$, so $|D| > |B/\mu_{\mathbf{B}}| \geq |B/\theta_i|$, for $i = 1, \dots, n$. Thus each $\mathbf{B}/\theta_i \in \mathcal{S}_0$, so $\mathbf{B}/\mu_{\mathbf{B}} \in \mathbf{V}(\mathcal{S}_0)$. It follows that $\mathbf{A}/\mu_{\mathbf{A}} \in \mathbf{V}(\mathcal{S}_0)$, and therefore $\mathbf{V}(\mathbf{A}/\mu_{\mathbf{A}})$ excludes \mathbf{A} . \square

Congruence-modularity is necessary to Theorem 2.8 as the next example shows.

Example 2.9. We recall Example 14 from [2]. Consider the algebra

$$\mathbf{A} = \langle \{0, 1, 2\}, \cdot, f, g, \underline{0}, \underline{1}, \underline{2} \rangle$$

of type $\langle 2, 1, 1, 0, 0, 0 \rangle$ in which

$$\begin{aligned} x \cdot y &= \begin{cases} 0, & \text{if } x = 0, \\ y, & \text{if } x \neq 0, \end{cases} \\ f(0) = f(1) &= 0, \quad f(2) = 2, \\ g(0) = g(2) &= 0, \quad g(1) = 1, \end{aligned}$$

and $\underline{0}$, $\underline{1}$, $\underline{2}$ are nullary operations naming the corresponding elements of A .

Let \mathcal{V} be the variety generated by \mathbf{A} . It is shown in [2] that \mathcal{V} is equationally complete. There is a congruence θ on $\mathbf{A} \times \mathbf{A}$ with a single non-trivial equivalence class: $\{(0, 0), (0, 1), (0, 2), (1, 0), (2, 0)\}$. Let $\mathbf{B} = \mathbf{A}^2/\theta$. The congruence lattice of \mathbf{B} is a 3-element chain: $0 < \mu < 1$. Certainly $\mathbf{B}/\mu \in \mathbf{V}(\mathbf{A})$, so by the equational completeness, $\mathbf{V}(\mathbf{B}/\mu) = \mathcal{V} = \mathbf{V}(\mathbf{B})$, showing that μ is not verbal on \mathbf{B} .

Theorem 2.8 also fails for infinite algebras. Indeed, the class of pseudo-simple algebras provides a rich class of such examples. An algebra \mathbf{A} is called *pseudo-simple* if it is nontrivial and if every nontrivial homomorphic image of \mathbf{A} is isomorphic to \mathbf{A} . Thus, the only verbal congruences of \mathbf{A} are 0_A and 1_A . As shown by D. Monk [16], every pseudo-simple algebra is subdirectly irreducible and has a congruence lattice that is either a 2-element chain or is order isomorphic to $\omega^\beta + 1$ for some ordinal β .

If \mathbf{A} is a pseudo-simple algebra whose congruence lattice is order isomorphic to $\omega + 1$, and if the monolith μ of \mathbf{A} is fully invariant, then Lemma 2.1(5) may be applied as in the proof of Theorem 2.3 to prove that every congruence relation of \mathbf{A} is fully invariant. For some familiar pseudo-simple algebras whose congruence lattices are order isomorphic to $\omega + 1$, Lemma 2.6 may be used to show that the monolith is fully invariant and therefore every one of its congruences is fully invariant, but none, except for 0 and 1, are verbal.

For example, the Prüfer group $\mathbb{Z}(p^\infty)$, p a prime, has congruence lattice $0 < \theta_1 < \theta_2 < \dots < 1$, and, for every positive integer n , $\mathbb{Z}(p^\infty)/\theta_n \cong \mathbb{Z}(p^\infty)$. So $\mathbb{Z}(p^\infty)$ is pseudo-simple. The monolith θ_1 is the principal congruence relation generated by $(0, \frac{1}{p})$. The elements 0 and $\frac{1}{p}$ and the term operation $f(x) = px$ satisfy the hypotheses of Lemma 2.6. Note that this algebra is subdirectly irreducible and generates a congruence-modular variety. Every one

of its congruences is fully invariant but none, except for 0 and 1, are verbal. Thus the finiteness condition of Theorem 2.8 is necessary.

Another pseudo-simple algebra \mathbf{A} for which all congruences are fully invariant but only 0_A and 1_A are verbal is the algebra $\langle \{0, 1, 2, \dots\}, f \rangle$ where f is a unary operation given by $f(0) = 0$ and $f(n) = n - 1$ for $n > 0$. Lemma 2.6, using 0, 1, and f , shows that the monolith is fully invariant.

With an eye towards investigating fully invariant congruences that are verbal, we present some facts about verbal congruences.

As stated in Definition 1.1, a congruence θ is verbal if there exists a variety \mathcal{V} such that $\theta = \lambda_{\mathcal{V}} = \bigcap \Lambda_{\mathcal{V}}$. If θ is verbal, then there may exist more than one variety \mathcal{W} for which $\theta = \lambda_{\mathcal{W}}$. However, we can always assert that θ is verbal if and only if $\theta = \lambda_{\mathbf{V}(\mathbf{A}/\theta)}$ and, if $\theta = \lambda_{\mathcal{W}}$ for some variety \mathcal{W} , then $\mathbf{A}/\theta \in \mathcal{W}$.

We commented earlier that full invariance does not behave well with respect to the formation of subalgebras, products, or homomorphic images. Verbal congruences do better under homomorphic images, as the next proposition shows, but just as badly under subalgebras or products. Corollary 2.11 can be seen as a counterpart to Lemma 2.1(5).

Proposition 2.10. *Let $f: \mathbf{A} \rightarrow \mathbf{B}$ be a surjective homomorphism with kernel θ . Then for any variety \mathcal{V} we have $\overleftarrow{f}(\lambda_{\mathcal{V}}^{\mathbf{B}}) = \lambda_{\mathcal{V}}^{\mathbf{A}} \vee \theta$.*

Proof. Let $\tau = \overleftarrow{f}(\lambda_{\mathcal{V}}^{\mathbf{B}})$ and $\psi = \lambda_{\mathcal{V}}^{\mathbf{A}} \vee \theta$. Then $\tau \geq \theta$ and $\mathbf{A}/\tau \cong \mathbf{B}/\lambda_{\mathcal{V}}^{\mathbf{B}} \in \mathcal{V}$, so $\tau \geq \lambda_{\mathcal{V}}^{\mathbf{A}}$. This shows $\tau \geq \psi$. On the other hand, let $\tau' = \overrightarrow{f}(\psi)$. Since $\psi \geq \lambda_{\mathcal{V}}^{\mathbf{A}}$, we see that $\mathbf{B}/\tau' \cong \mathbf{A}/\psi \in \mathcal{V}$. Thus $\tau' \geq \lambda_{\mathcal{V}}^{\mathbf{B}}$. Consequently, $\psi \geq \tau$. \square

Corollary 2.11. *Suppose $\theta \leq \tau \in \text{Con } \mathbf{A}$. If θ is verbal on \mathbf{A} , and τ/θ is verbal on \mathbf{A}/θ , then τ is verbal on \mathbf{A} .*

Proof. Let $\mathbf{B} = \mathbf{A}/\theta$ and $f: \mathbf{A} \rightarrow \mathbf{B}$ be the canonical map. By assumption, θ and $\tau/\theta = \overrightarrow{f}(\tau)$ are verbal, i.e., $\theta = \lambda_{\mathcal{V}}^{\mathbf{A}}$ and $\overrightarrow{f}(\tau) = \lambda_{\mathcal{W}}^{\mathbf{B}}$ for varieties \mathcal{V} and \mathcal{W} . By Proposition 2.10, $\tau = \overleftarrow{f}(\overrightarrow{f}(\tau)) = \lambda_{\mathcal{W}}^{\mathbf{A}} \vee \theta = \lambda_{\mathcal{V} \wedge \mathcal{W}}^{\mathbf{A}}$ by the formula $\lambda_{\mathcal{V}} \vee \lambda_{\mathcal{W}} = \lambda_{\mathcal{V} \wedge \mathcal{W}}$ of display (1). \square

We promised an example to show that the inclusion in (1)

$$\lambda_{\mathcal{V}} \wedge \lambda_{\mathcal{W}} \supseteq \lambda_{\mathcal{V} \vee \mathcal{W}}$$

can be sharp. Here it is.

Example 2.12. Let s denote the unary identity operation and r denote complementation on $\{0, 1\}$. Our similarity type will have two unary operation symbols, f and g . Define the algebras

$$\begin{aligned} \mathbf{A} &= \langle \{0, 1\}, s, r \rangle \\ \mathbf{B} &= \langle \{0, 1\}, r, s \rangle. \end{aligned}$$

Let θ be the congruence on $\mathbf{A} \times \mathbf{B}$ that identifies $(0, 0)$ with $(1, 1)$ and $(0, 1)$ with $(1, 0)$. Let $\mathbf{C} = (\mathbf{A} \times \mathbf{B})/\theta$. The algebra \mathbf{C} has two elements, so it is simple.

We observe that $\mathcal{V} = \mathbf{V}(\mathbf{A})$ satisfies the identity $f(x) \approx x$, while $\mathcal{W} = \mathbf{V}(\mathbf{B})$ satisfies $g(x) \approx x$. Clearly, $\mathbf{C} \in \mathcal{V} \vee \mathcal{W}$, so $\lambda_{\mathcal{V} \vee \mathcal{W}}^{\mathbf{C}} = 0_{\mathbf{C}}$. But \mathbf{C} fails to satisfy the identities defining either \mathcal{V} or \mathcal{W} , so, since \mathbf{C} is simple, we must have $\lambda_{\mathcal{V}}^{\mathbf{C}} = \lambda_{\mathcal{W}}^{\mathbf{C}} = 1_{\mathbf{C}}$. Consequently, $\lambda_{\mathcal{V}}^{\mathbf{C}} \wedge \lambda_{\mathcal{W}}^{\mathbf{C}} \not\geq \lambda_{\mathcal{V} \vee \mathcal{W}}^{\mathbf{C}}$.

Here is a simple observation. Suppose that $\sigma, \tau, \theta \in \text{Con } \mathbf{A}$, θ is verbal and $\mathbf{A}/\sigma \cong \mathbf{A}/\tau$. Then $\tau \geq \theta$ implies $\sigma \geq \theta$. In order to utilize this principle we introduce some notation.

Definition 2.13. For algebras \mathbf{A} and \mathbf{B} define

$$\Delta(\mathbf{A}, \mathbf{B}) = \{\tau \in \text{Con } \mathbf{A} : \mathbf{A}/\tau \cong \mathbf{B}\} \text{ and } \delta_{\mathbf{B}} = \bigwedge \Delta(\mathbf{A}, \mathbf{B}).$$

Furthermore, for a set \mathcal{T} of algebras let $\delta_{\mathcal{T}}$ denote $\bigwedge \{\delta_{\mathbf{B}} : \mathbf{B} \in \mathcal{T}\}$.

If $\theta \in \text{Con } \mathbf{A}$ is verbal, then for $\tau \in \text{Con } \mathbf{A}$ either $\theta \leq \delta_{\mathbf{A}/\tau}$ or $\theta \not\leq \tau$. An alternate phrasing of this observation in terms of subdirectly irreducible algebras is that if θ is a verbal congruence, then $\theta = \delta_{\mathcal{T}}$, where \mathcal{T} is the set of subdirectly irreducible homomorphic images of \mathbf{A}/θ . Thus, if a fully invariant congruence relation, θ , of \mathbf{A} , were to be verbal, then $\theta \leq \delta_{\mathbf{A}/\tau}$ for every $\tau \geq \theta$ and $\theta = \bigwedge \{\delta_{\mathbf{A}/\gamma} : \gamma \geq \theta, \gamma \text{ completely meet irreducible}\}$. The next lemma shows that if $\theta \in \text{Con } \mathbf{A}$ is fully invariant, and if $\tau \geq \theta$ is such that \mathbf{A}/τ is a retract of \mathbf{A} , then $\theta \leq \delta_{\mathbf{A}/\tau}$.

Definition 2.14. An algebra \mathbf{B} is a *retract* of an algebra \mathbf{A} if there is a homomorphism $f: \mathbf{B} \rightarrow \mathbf{A}$ and a homomorphism $r: \mathbf{A} \rightarrow \mathbf{B}$ such that rf is the identity on \mathbf{B} . The homomorphism r is called a *retraction*. The map f is necessarily one-to-one and r is onto.

Suppose that r is a retraction of \mathbf{A} on \mathbf{B} as in the definition. Let $r' = f \circ r$. Then r' is an endomorphism of \mathbf{A} and r' is idempotent, that is, $r' \circ r' = r'$. Conversely, any idempotent endomorphism, r' , is a retraction of \mathbf{A} onto the image of r' with the inclusion map as the one-sided inverse.

Lemma 2.15. *If r is a retraction of \mathbf{A} on \mathbf{B} and if θ is a fully invariant congruence of \mathbf{A} with $\theta \leq \ker r$, then $\theta \leq \delta_{\mathbf{B}}$.*

Proof. Since r is a retraction, let $f: \mathbf{B} \rightarrow \mathbf{A}$ be a homomorphism for which $r \circ f = \text{id}_{\mathbf{B}}$. Consider any homomorphism $s: \mathbf{A} \rightarrow \mathbf{B}$. Let $\bar{s} = f \circ s$. Then \bar{s} is an endomorphism of \mathbf{A} . Now suppose that θ is fully invariant and $\theta \leq \ker r$. Then for all $a, b \in \mathbf{A}$,

$$\begin{aligned} (a, b) \in \theta &\implies (\bar{s}(a), \bar{s}(b)) \in \theta \subseteq \ker r \implies \\ r\bar{s}(a) &= r\bar{s}(b) \implies rfs(a) = rfs(b) \implies s(a) = s(b). \end{aligned}$$

Thus $\theta \leq \ker s$. □

3. Verbose algebras

We now turn to verbose algebras, that is, algebras in which every fully invariant congruence is verbal. Obviously, every simple algebra is verbose. More generally, we have the following observation.

Theorem 3.1. *Let \mathbf{A} be a finite algebra in a congruence-modular variety, and suppose that $\text{Con } \mathbf{A}$ forms a chain. Then \mathbf{A} is verbose.*

Proof. It follows from Corollary 2.4 that every congruence is fully invariant. So we must show that every congruence is verbal. The proof is by induction on $n = |\text{Con } \mathbf{A}|$. If $n \leq 2$ then \mathbf{A} is either trivial or simple. In either case, every congruence is verbal. So assume the theorem holds for all finite algebras whose congruence lattice is a chain of size less than n . By Theorem 2.8, $\mu_{\mathbf{A}}$ is verbal and by the induction hypothesis, $\mathbf{A}/\mu_{\mathbf{A}}$ is verbose. An application of Corollary 2.11 now shows that every congruence on \mathbf{A} is verbal. \square

Example 2.9 shows that Theorem 3.1 fails without congruence-modularity and the Prüfer groups show the necessity of finiteness.

A second important class of verbose algebras are the projectives.

Definition 3.2. An algebra \mathbf{P} in a class \mathcal{K} of algebras is called *projective in \mathcal{K}* if for any homomorphism $h: \mathbf{P} \rightarrow \mathbf{B}$ and surjection $g: \mathbf{A} \rightarrow \mathbf{B}$ with $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, there exists a homomorphism $f: \mathbf{P} \rightarrow \mathbf{A}$ such that $h = g \circ f$.

The following characterizations of projective algebras are standard.

Proposition 3.3. *Let \mathcal{K} be a class of algebras for which \mathcal{K} -free algebras exist. Then for any $\mathbf{P} \in \mathcal{K}$ the following conditions are equivalent.*

- (1) \mathbf{P} is projective in \mathcal{K} .
- (2) For every $\mathbf{A} \in \mathcal{K}$ and surjective homomorphism $r: \mathbf{A} \rightarrow \mathbf{P}$ there is a homomorphism $f: \mathbf{P} \rightarrow \mathbf{A}$ such that $r \circ f$ is the identity on \mathbf{P} . (That is, \mathbf{P} is a retract of \mathbf{A} and r is a retraction.)
- (3) \mathbf{P} is a retract of a \mathcal{K} -free algebra.

It follows from Proposition 3.3 that every free algebra is projective. Thus the following theorem generalizes Proposition 1.4.

Theorem 3.4. *If \mathbf{P} is projective in $\mathbf{V}(\mathbf{P})$, then \mathbf{P} is verbose.*

Proof. Let \mathbf{P} be projective. Then \mathbf{P} is a retract of a free algebra. That is, there is a set X and an algebra \mathbf{F} , free on X , and an endomorphism r of \mathbf{F} such that \mathbf{P} is the image of r and $r \circ r = r$.

Let θ be a congruence on \mathbf{F} with $\mathbf{B} = \mathbf{F}/\theta$ and \mathcal{W} the variety generated by \mathbf{B} .

Claim. *Let $(a, c) \in \theta$ and suppose that for every $g \in \text{End}(\mathbf{F})$ we have $(g(a), g(c)) \in \theta$. Then $(a, c) \in \lambda_{\mathcal{W}}^{\mathbf{F}}$.*

Proof: There are $x_1, \dots, x_n \in X$ and terms p and q with $a = p^{\mathbf{F}}(x_1, \dots, x_n)$ and $c = q^{\mathbf{F}}(x_1, \dots, x_n)$. We first show that $\mathbf{B} \models p \approx q$. Let $b_1, \dots, b_n \in B$ and pick $b'_i \in F$ such that $b'_i/\theta = b_i$, for $i \leq n$. From the freeness of \mathbf{F} , there is an endomorphism g such that $g(x_i) = b'_i$, for $i \leq n$. Then

$$p^{\mathbf{B}}(\mathbf{b}) = p^{\mathbf{F}}(\mathbf{b}')/\theta = g(p^{\mathbf{F}}(\mathbf{x}))/\theta = g(a)/\theta$$

and similarly, $q^{\mathbf{B}}(\mathbf{b}) = g(c)/\theta$. Since our assumption is that $g(a) \theta g(c)$, we conclude that $p(\mathbf{b}) = q(\mathbf{b})$. Thus $\mathbf{B} \models p \approx q$.

Since \mathbf{B} is generic for \mathcal{W} , we must have $\mathcal{W} \models p \approx q$. Also, $\mathbf{A} = \mathbf{F}/\lambda_{\mathcal{W}}^{\mathbf{F}}$ is a member of \mathcal{W} , so $a/\lambda_{\mathcal{W}}^{\mathbf{F}} = p^{\mathbf{A}}(\mathbf{x}/\lambda_{\mathcal{W}}^{\mathbf{F}}) = q^{\mathbf{A}}(\mathbf{x}/\lambda_{\mathcal{W}}^{\mathbf{F}}) = c/\lambda_{\mathcal{W}}^{\mathbf{F}}$, proving the claim.

Now we prove the theorem. Let θ be a congruence on \mathbf{P} and assume that θ is not verbal. Let $\mathbf{B} = \mathbf{P}/\theta$ and \mathcal{W} the variety generated by \mathbf{B} . Then $\lambda_{\mathcal{W}}^{\mathbf{P}} < \theta$. Fix $(a, c) \in \theta \setminus \lambda_{\mathcal{W}}^{\mathbf{P}}$. Let $\bar{\theta} = \check{\mathcal{F}}(\theta)$. Then $\lambda_{\mathcal{W}}^{\mathbf{P}} \leq \check{\mathcal{F}}(\lambda_{\mathcal{W}}^{\mathbf{P}}) \leq \bar{\theta}$, so $(a, c) \in \theta \setminus \lambda_{\mathcal{W}}^{\mathbf{P}}$. Therefore by the claim, there must be an endomorphism g of \mathbf{F} such that $(g(a), g(c)) \notin \bar{\theta}$. Let $e = r \circ g \upharpoonright_P \in \text{End}(\mathbf{P})$. Then $(g(a), g(c)) \notin \bar{\theta}$ implies that $(e(a), e(c)) = (rg(a), rg(c)) \notin \theta$. Thus θ is not fully invariant. \square

Subdirectly irreducible, projective algebras turn out to be useful in proving algebras verbose. The following somewhat technical theorem will have applications in the next section.

Theorem 3.5. *Let \mathbf{A} be an algebra with these two properties:*

- (1) *If $\mathbf{S} \in (\mathbf{H}(\mathbf{A}))_{\text{si}}$, then \mathbf{S} is projective in $\mathbf{V}(\mathbf{A})$.*
- (2) *If $\mathbf{S} \in (\mathbf{H}(\mathbf{A}))_{\text{si}}$ and $\mathcal{T} \subseteq (\mathbf{H}(\mathbf{A}))_{\text{si}}$ with $\mathbf{S} \in \mathbf{V}(\mathcal{T})$, then $\mathbf{S} \in \mathbf{HS}(\mathcal{T})$.*

Then \mathbf{A} is verbose.

Proof. Let $\theta \in \text{Con } \mathbf{A}$ be a fully invariant congruence relation. We wish to show θ is verbal. We may assume $\theta \neq 0_A, 1_A$.

Let $\mathcal{T} = (\mathbf{H}(\mathbf{A}/\theta))_{\text{si}}$. Each $\mathbf{B} \in \mathcal{T}$ is projective in $\mathbf{V}(\mathbf{A})$ by hypothesis and is therefore a retract of \mathbf{A}/θ . Thus $\theta = \delta_{\mathcal{T}}$ by Lemma 2.15. Let \mathcal{W} be the variety generated by \mathcal{T} . We shall show that if $\mathbf{C} \in \mathcal{W}_{\text{si}} \cap \mathbf{H}(\mathbf{A})$, then $\mathbf{C} \in \mathcal{T}$. From this it follows that $\lambda_{\mathcal{W}} = \delta_{\mathcal{W}} = \delta_{\mathcal{W}_{\text{si}} \cap \mathbf{H}(\mathbf{A})} = \delta_{\mathcal{T}} = \theta$.

We have $\mathbf{C} \in \mathbf{HS}(\mathcal{T})$ by hypothesis (2). So there is an algebra $\mathbf{B} \in \mathcal{T}$ and a subalgebra \mathbf{D} of \mathbf{B} such that \mathbf{C} is a homomorphic image of \mathbf{D} . Since \mathbf{C} is subdirectly irreducible, it is projective in \mathcal{W} . As a homomorphic image of \mathbf{D} , it is a retract. Hence there is an embedding $g : \mathbf{C} \rightarrow \mathbf{B}$.

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{g_1} & \mathbf{B} \\ \downarrow & & \\ \mathbf{C} & & \end{array} \quad g = g_1 \circ g_2$$

Since \mathbf{C} is a subdirectly irreducible homomorphic image of \mathbf{A} , there is a congruence $\beta < 1_A$ such that $\mathbf{A}/\beta \cong \mathbf{C}$. Let s denote the composite map $\mathbf{A} \rightarrow \mathbf{C} \xrightarrow{g} \mathbf{B}$ with kernel β . Since $\mathbf{B} \in \mathcal{T}$, there is a surjective map $r : \mathbf{A} \rightarrow \mathbf{B}$

with $\theta \leq \ker r$. Also, \mathbf{B} is subdirectly irreducible, hence projective, so r is a retraction. Therefore by Lemma 2.15, $\theta \leq \ker s$, that is, $\mathbf{C} \in \mathcal{T}$ as desired. \square

As an application of these principles, let us examine finite Abelian groups.

Lemma 3.6. *Let \mathbf{A}_1 and \mathbf{A}_2 be finite Abelian groups of orders r_1 and r_2 , respectively. If r_1 and r_2 are relatively prime, then*

$$\begin{aligned} \text{Con}(\mathbf{A}_1 \times \mathbf{A}_2) &= \text{Con}(\mathbf{A}_1) \times \text{Con}(\mathbf{A}_2) \\ \text{End}(\mathbf{A}_1 \times \mathbf{A}_2) &= \text{End}(\mathbf{A}_1) \times \text{End}(\mathbf{A}_2). \end{aligned} \tag{2}$$

Consequently, a congruence $\theta = \theta_1 \times \theta_2$ on $\mathbf{A}_1 \times \mathbf{A}_2$ is fully invariant (resp. verbal) iff both θ_1 and θ_2 are fully invariant (resp. verbal).

Proof. Since r_1 and r_2 are relatively prime, there are integers s_1, s_2 such that $r_1 s_1 + r_2 s_2 = 1$. Let $f(x_1, x_2) = r_2 s_2 x_1 + r_1 s_1 x_2$. Then $\mathbf{A}_i \models f(x_1, x_2) \approx x_i$. One can now use elementary group theory to verify the relationships in (2). Alternately, we have shown that f is a decomposition term for $\mathbf{V}(\mathbf{A}_1 \times \mathbf{A}_2)$. Equations (2) follow from this observation, see [15, Theorem 4.36]. \square

Since every finite Abelian group is a direct product of its Sylow subgroups, the above lemma, applied inductively to the number of distinct prime factors, yields the following theorem.

Theorem 3.7. *A finite Abelian group is verbose if and only if each of its Sylow subgroups is verbose.*

Thus the characterization of finite, verbose Abelian groups is reduced to the case of finite Abelian p -groups.

Theorem 3.8. *Let p be a prime and let \mathbf{A} be a finite Abelian p -group. Then \mathbf{A} is verbose if and only if it is a direct power of a cyclic group.*

Proof. If \mathbf{A} is a power of a cyclic group, then it is of the form $(\mathbb{Z}_{p^n})^m$. Thus \mathbf{A} is free (on m generators) in the variety $\mathbf{V}(\mathbf{A})$. By Theorem 3.4, \mathbf{A} is verbose.

Now assume that \mathbf{A} is not of the desired form. Then, \mathbf{A} must be a direct sum of cyclic subgroups, not all of which have the same order. That is, we can write

$$\mathbf{A} = \langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \cdots \oplus \langle a_m \rangle \oplus \langle b_1 \rangle \oplus \cdots \oplus \langle b_n \rangle \oplus \langle c_1 \rangle \oplus \cdots \oplus \langle c_q \rangle$$

in which every a_i has order p^ℓ , every b_i has order p^k , $\ell > k$, every c_i has order less than p^k , $m, n \geq 1$, and $q \geq 0$.

For this argument, it is easier to work with subgroups rather than congruences. Let

$$S = \{x \in A : p^k \cdot x = 0\} \text{ and } T = \{p^{\ell-k} \cdot x : x \in A\}.$$

Both S and T are easily seen to be subgroups of \mathbf{A} . Also, $T \subseteq S$ since if $y \in T$ then $y = p^{\ell-k}x$, so $p^k y = p^\ell x = 0$, since \mathbf{A} has exponent p^ℓ . On the other hand, $b_1 \in S \setminus T$ so T is a proper subgroup of S .

S is a fully invariant subgroup, for if $x \in S$ and e is an endomorphism, then $p^k \cdot e(x) = e(p^k x) = e(0) = 0$. But S is not verbal. To see this, we compute

$$\mathbf{A}/S = \langle a_1 \rangle / \langle p^{\ell-k} a_1 \rangle \oplus \cdots \oplus \langle a_m \rangle / \langle p^{\ell-k} a_m \rangle \cong (\mathbb{Z}_{p^{\ell-k}})^m$$

while

$$\begin{aligned} \mathbf{A}/T &= \langle a_1 \rangle / \langle p^{\ell-k} a_1 \rangle \oplus \cdots \oplus \langle a_m \rangle / \langle p^{\ell-k} a_m \rangle \oplus \langle b_1 \rangle / \langle p^{\ell-k} b_1 \rangle \oplus \cdots \\ &\quad \oplus \langle b_n \rangle / \langle p^{\ell-k} b_n \rangle \oplus \langle c_1 \rangle / \langle p^{\ell-k} c_1 \rangle \oplus \cdots \cong (\mathbb{Z}_{p^{\ell-k}})^{m+n} \oplus \mathbf{B} \end{aligned}$$

in which \mathbf{B} has exponent strictly dividing $p^{\ell-k}$. It follows that \mathbf{A}/S and \mathbf{A}/T generate the same variety and T is smaller than S . Thus \mathbf{A} fails to be verbose. \square

4. Verbose varieties

The next step in our logical progression is the consideration of varieties in which every member is verbose. These results are not surprising. Many varieties with nice structure turn out to be verbose.

Example 4.1. We continue our discussion of Abelian groups. Let \mathcal{A} denote the variety of all Abelian groups and, for each natural number n , \mathcal{A}_n the subvariety defined by the identity $nx \approx 0$. Note that $\mathcal{A}_0 = \mathcal{A}$ and \mathcal{A}_1 is the trivial variety. $\{\mathcal{A}_n : n \in \omega\}$ constitutes the entire lattice of subvarieties of \mathcal{A} (see [1, Theorem 4.46]). We continue to use the conventions of Theorem 3.8.

Suppose first that n is square-free. The subdirectly irreducible members of \mathcal{A}_n are the cyclic groups of prime order, p , where p divides n , while the free algebra on one generator is the cyclic group of order n . It is easy to see that \mathbb{Z}_p is a retract of \mathbb{Z}_n (since n is square-free), so by Proposition 3.3, \mathbb{Z}_p is projective in \mathcal{A}_n . It is now easy to apply Theorem 3.5 to show that \mathcal{A}_n is verbose.

On the other hand, suppose that n is either 0 or is not square-free. Then for some prime p , \mathcal{A}_n contains a group isomorphic to $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$. By Theorem 3.8, this group, hence this variety, fails to be verbose. Thus the verbose varieties of Abelian groups are precisely the varieties of nonzero, square-free exponent.

Theorem 4.2. *Let \mathcal{V} be a finitely generated, congruence-distributive variety and suppose that every subdirectly irreducible member of \mathcal{V} is projective in \mathcal{V} . Then \mathcal{V} is verbose.*

Proof. The variety \mathcal{V} has only finitely many subdirectly irreducible algebras and all are finite since \mathcal{V} is finitely generated, By Jónsson's Lemma, if $\mathcal{T} \subseteq \mathcal{V}_{\text{si}}$ and $\mathbf{S} \in (\mathbf{V}(\mathcal{T}))_{\text{si}}$, then $\mathbf{S} \in \mathbf{HS}(\mathcal{T})$. Thus, Theorem 3.5 applies to every $\mathbf{A} \in \mathcal{V}$. \square

The converse of Theorem 4.2 is false. There are finitely generated, congruence-distributive, verbose varieties containing subdirectly irreducible algebras

which are not projective. See Example 5.5. We have been unable to determine whether the theorem can be extended even to congruence modular varieties.

Theorem 4.3. *Let \mathcal{V} be any variety containing exactly one subdirectly irreducible algebra \mathbf{S} , and suppose that \mathbf{S} is finite. Then \mathcal{V} is verbose.*

Proof. Since \mathcal{V} is a minimal variety, it suffices to show that for any $\mathbf{A} \in \mathcal{V}$ the only fully invariant congruences of \mathbf{A} are 0_A and 1_A . Clearly \mathcal{V} is generated by \mathbf{S} , is locally finite, and \mathbf{S} is strictly simple. Hence \mathbf{S} is projective in \mathcal{V} by a result of K. Kearnes and Á. Szendrei [9, Theorem 3.2]. Therefore by Lemma 2.15, a fully invariant congruence on \mathbf{A} is equal to either 1_A or to $\delta_{\mathbf{S}} = 0_A$. \square

Corollary 4.4. *Let \mathcal{V} be a locally finite, congruence-modular, minimal variety. Then \mathcal{V} is verbose.*

Proof. Let \mathbf{S} be a strictly simple member of \mathcal{V} . By [2, Lemma 4] \mathbf{S} is the unique subdirectly irreducible algebra in \mathcal{V} . So \mathcal{V} is verbose by Theorem 4.3. \square

We now present some examples to illustrate the results of this section.

Example 4.5. The 2-element subdirectly irreducible lattice \mathbf{D} is easily seen to be projective in the variety of all lattices. Hence the variety of distributive lattices, which is generated by \mathbf{D} , is verbose by Theorem 4.2. The 5-element nonmodular lattice \mathbf{N}_5 is also known [6] to be projective in the variety of all lattices. Thus, the variety $\mathbf{V}(\mathbf{N}_5)$ is verbose.

For $n \geq 3$, let \mathbf{M}_n denote the $(n+2)$ -element lattice of length 2. If \mathcal{V} is any locally finite variety of lattices that contains \mathbf{M}_n , then \mathbf{M}_n is projective in \mathcal{V} . To see this let $M_n = \{0, 1, a_1, \dots, a_n\}$, let $\{x_1, \dots, x_n\}$ denote the free generators of $\mathbf{F}_{\mathcal{V}}(n)$, and let $h: \mathbf{F}_{\mathcal{V}}(n) \rightarrow \mathbf{M}_n$ denote the canonical homomorphism given by $h(x_i) = a_i$. The lattice $\mathbf{F}_{\mathcal{V}}(n)$ is finite. Let v and u in $\mathbf{F}_{\mathcal{V}}(n)$ denote $\bigwedge \overleftarrow{h}(1)$ and $\bigvee \overleftarrow{h}(0)$, respectively. Consider $p_i = (x_i \vee u) \wedge v$ for $1 \leq i \leq n$. Then $u < p_i < v$ and $\{u, v, p_1, \dots, p_n\}$ is the universe of a sublattice of $\mathbf{F}_{\mathcal{V}}(n)$ that is isomorphic to \mathbf{M}_n , and the image of this sublattice under h is \mathbf{M}_n . So \mathbf{M}_n is a retract of $\mathbf{F}_{\mathcal{V}}(n)$ and is thus a projective subdirectly irreducible algebra in \mathcal{V} .

Therefore, applying Theorem 4.2 to the comments in the previous two paragraphs we conclude that the variety \mathcal{V} generated by \mathbf{N}_5 and \mathbf{M}_n is verbose since the subdirectly irreducible algebras in this variety are \mathbf{D} , \mathbf{N}_5 , \mathbf{M}_3 , \dots , \mathbf{M}_n , which are all projective in \mathcal{V} .

Example 4.5 demonstrates that several varieties of lattices are verbose. The proof relies on Theorem 4.2 and, consequently, on the fact that every subdirectly irreducible member of the variety is projective. The following example demonstrates the importance of projectivity. The variety $\mathbf{V}(\mathbf{M}_3)$ is known from [8] to have precisely three covers: $\mathbf{V}(\mathbf{M}_4)$, $\mathbf{V}(\mathbf{M}_3, \mathbf{N}_5)$ and $\mathbf{V}(\mathbf{M}_{3,3})$. The first two of these are verbose by Example 4.5. Let us consider the last one.

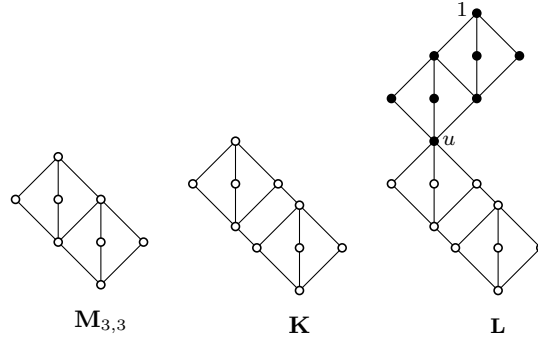


FIGURE 2

Example 4.6. The variety $\mathbf{V}(M_{3,3})$ is not verbose. The lattice $M_{3,3}$ is shown in Figure 2. That figure shows two additional lattices, L and K . The lattice K is a subdirect product of $M_{3,3}$ and D (the 2-element lattice), while L is a subdirect product of $M_{3,3}$ and K . From this we see that $L \in \mathbf{V}(M_{3,3})$. Since $M_{3,3}$ is a homomorphic image of, but not a sublattice of K , we conclude that $M_{3,3}$ fails to be projective in its own variety.

Let θ be the congruence on L generated by the pair $(u, 1)$. This congruence has one nontrivial class, C , namely the interval between u and 1 . θ is not verbal since $\theta \neq 0_L$ but $M_{3,3} \in \mathbf{H}(L/\theta)$.

However θ is fully invariant. Suppose e is an endomorphism of L and $(a, b) \in \theta$. If $a = b$ then of course $(e(a), e(b)) \in \theta$. So assume that a and b are distinct. Then they must be members of C . But note that $C \cong M_{3,3}$ is a simple lattice. Thus either $e(a) = e(b)$, in which case we are done, or e is one-to-one on C . But since L has only one sublattice isomorphic to $M_{3,3}$, the endomorphism e must map C onto itself. Thus $(e(a), e(b)) \in \theta$.

In Corollary 4.4 we considered minimal, congruence-modular varieties. We can extend that result as follows.

Theorem 4.7. *Let \mathbf{A} be a finite simple algebra generating a congruence-modular, Abelian variety, \mathcal{V} . Then \mathcal{V} is verbose.*

Proof. Let us first make the following observation. For any algebra \mathbf{A} and positive integers $m \leq n$, the algebra \mathbf{A}^m is a retract of \mathbf{A}^n . For the embedding of \mathbf{A}^m into \mathbf{A}^n we can map (a_1, \dots, a_m) to $(a_1, a_2, \dots, a_m, a_m, \dots, a_m)$. The projection of \mathbf{A}^n onto its first m coordinates is a one-sided inverse of this embedding.

For the proof of the theorem, we rely on the analysis in [7, Theorem 12.4]. An element a in an algebra \mathbf{A} is called *idempotent* if, for every basic operation f , $f(a, a, \dots, a) = a$. Suppose first that \mathbf{A} has an idempotent element. Then every finite algebra in \mathcal{V} is isomorphic to a direct power of \mathbf{A} . Moreover, for every infinite cardinal κ , there is, up to isomorphism, a unique member of \mathcal{V} of cardinality κ .

Since the finitely generated free algebras must get arbitrarily large, for every m , there is an $n \geq m$ such that \mathbf{A}^n is free. Using the observation in the opening paragraph, \mathbf{A}^m is a retract of the free algebra \mathbf{A}^n . Therefore, by Proposition 3.3 and Theorem 3.4, every finite algebra is verbose.

For the infinite members of \mathcal{V} , let κ be an infinite cardinal. Since there is a unique algebra of cardinality κ , it must be free, hence verbose.

Now assume that \mathbf{A} has no idempotent element. The argument is almost the same as above. In this case, \mathcal{V} has two subdirectly irreducible algebras, \mathbf{A} and $\mathbf{B} = \mathbf{A}_\nabla$. \mathbf{B} is a quotient of \mathbf{A}^2 , is simple, has an idempotent element, and generates the unique proper nontrivial subvariety of \mathcal{V} . Thus the argument in the previous two paragraphs applies to $\mathbf{V}(\mathbf{B})$, showing that this subvariety is verbose.

Now according to [7], every finite algebra in $\mathcal{V} \setminus \mathbf{V}(\mathbf{B})$ is of the form \mathbf{A}^m for some m . Choose a sufficiently large finite \mathcal{V} -free algebra \mathbf{F} such that $|F| > |A|^m$ and $\mathbf{F} \notin \mathbf{V}(\mathbf{B})$. The algebra \mathbf{F} will be a power of \mathbf{A} , so again using our initial observation, \mathbf{A}^m is a retract of \mathbf{F} . Thus \mathbf{A}^m is projective, hence verbose. Finally, for every infinite κ , there is a unique algebra in $\mathcal{V} \setminus \mathbf{V}(\mathbf{B})$ of cardinality κ . This algebra must be free, hence verbose. \square

Example 4.8. We examine each variety \mathcal{V} that is generated by a 2-element algebra. We use the classification of the 2-element algebras given by E. Post [17]. If \mathcal{V} is congruence-distributive, then \mathcal{V} has only one subdirectly irreducible member, so by Theorem 4.3, it is verbose. If \mathcal{V} is congruence-permutable, but not congruence-distributive, then \mathcal{V} is, essentially, one of four varieties: Boolean groups, Boolean 3-groups, complemented Boolean groups, or complemented Boolean 3-groups. Each of these varieties is congruence-modular, Abelian, and is generated by a simple algebra. So these four varieties are all verbose by Theorem 4.7.

An examination of Post's classification reveals that the only remaining 2-element algebras to consider are the following.

- The 2-element semilattice;
- the unary algebra $\langle \{0, 1\}, r \rangle$ with complementation: $r(0) = 1, r(1) = 0$;
- the 2-element set.

In each case, there is the possibility that the algebra also has either one or two nullary operations.

It follows from Theorem 4.3 that the varieties generated by a 2-element semilattice with or without constants are all verbose since there is only one subdirectly irreducible algebra.

Next, we consider the variety \mathcal{V} generated by a 2-element algebra having no fundamental operations except for zero, one or two constant operations. If $\mathbf{A} \in \mathcal{V}$, then every equivalence relation on A is a congruence relation of \mathbf{A} . Similarly, a function $e: A \rightarrow A$ is an endomorphism if and only if $e(c) = c$ for every constant term c . If our similarity type has at most one nullary symbol,

then this variety is equationally complete and it is easy to see that no proper nontrivial equivalence relation can be fully invariant.

So suppose we have two nullary operation symbols, call them c and d . The variety \mathcal{V} is generated by $\langle\{0, 1\}, 0, 1\rangle$ and has one proper subvariety, generated by $\mathbf{C} = \langle\{0, 1\}, 0, 0\rangle$ (and satisfying the identity $c \approx d$). For an algebra $\mathbf{A} \in \mathcal{V}$, in addition to 0_A and 1_A , the only other verbal congruence relation on \mathbf{A} is $\delta_{\mathbf{C}}$, which identifies $c^{\mathbf{A}}$ with $d^{\mathbf{A}}$, and nothing else. If θ is any congruence relation of \mathbf{A} that is not verbal, then there are distinct elements $x, y, z \in A$ such $(x, y) \in \theta$, $(y, z) \notin \theta$, and x is neither c nor d . Let $e: A \rightarrow A$ be any map such that $e(y) = y$ and $e(x) = z$. Then e is an endomorphism of \mathbf{A} that witnesses θ not being fully invariant. This shows that \mathcal{V} is verbose.

Finally, it remains to consider the 2-element unary algebra where the fundamental operation is complementation. Consider the variety \mathcal{V} generated by the algebra $\mathbf{2} = \langle\{0, 1\}, r\rangle$. The variety has two other subdirectly irreducible algebras: $\mathbf{2} + \mathbf{1}$ and $\mathbf{1} + \mathbf{1}$, where $\mathbf{1}$ denotes the trivial algebra in \mathcal{V} and $+$ denotes disjoint union. We have $\mathbf{V}(\mathbf{2}) = \mathbf{V}(\mathbf{2} + \mathbf{1}) > \mathbf{V}(\mathbf{1} + \mathbf{1})$. The algebra $\mathbf{2}$ is simple and it is projective since it is free.

Consider the algebra $\mathbf{A} = \{a_0, a_1\} \cup \{b\} \cup \{c\} \cong \mathbf{2} + \mathbf{1} + \mathbf{1}$. The only proper nontrivial verbal congruence is $\delta_{\mathbf{1}+\mathbf{1}}$, which identifies a_0 and a_1 . Let θ be the congruence on \mathbf{A} generated by $\{(a_0, a_1), (b, c)\}$. Since $\delta_{\mathbf{1}+\mathbf{1}} < \theta < 1$, θ is not verbal. We show θ is fully invariant. Let e be any endomorphism of \mathbf{A} . Since b and c are fixed points of r , it follows that $\{e(b), e(c)\} \subseteq \{b, c\}$. Likewise the set $\{e(a_0), e(a_1)\}$ is either $\{a_0, a_1\}$, $\{b\}$, or $\{c\}$. So $e(\theta) \subseteq \theta$. Thus θ is fully invariant but not verbal. The addition of constant symbols does not affect this example.

In summary, every 2-element algebra generates a verbose variety, except for $\langle 2, r \rangle$, $\langle 2, r, 0 \rangle$, $\langle 2, r, 1 \rangle$ and $\langle 2, r, 0, 1 \rangle$.

5. Arithmetical Semisimple Varieties

The analysis in Section 4 shows that, while useful, projectivity is not the key to understanding verbosity, even for congruence-distributive varieties. In this section we specialize further still to obtain some positive results.

Let \mathcal{V} be a finitely generated, congruence-distributive variety, and let $\mathbf{A} \in \mathcal{V}$. We write $I^{\mathbf{A}}$ for the set of completely meet-irreducible congruences on \mathbf{A} . We shall usually omit the superscript on I . Define a binary relation on I by

$$\alpha \sqsubseteq \beta \iff \mathbf{A}/\alpha \in \mathbf{HS}(\mathbf{A}/\beta).$$

Note that \sqsubseteq is a quasiorder on I and $\alpha \sqsubseteq \beta \sqsubseteq \alpha$ if and only if $\mathbf{A}/\alpha \cong \mathbf{A}/\beta$. Furthermore, for an arbitrary congruence, θ , let $I(\theta) = \{\alpha \in I : \theta \leq \alpha\}$.

Lemma 5.1. *The congruence θ is verbal if and only if $I(\theta)$ is a downset under \sqsubseteq .*

Proof. Suppose first that θ is verbal and that $\beta \sqsubseteq \alpha \in I(\theta)$. Then $\theta \leq \alpha$, so $\mathbf{A}/\alpha \in \mathbf{V}(\mathbf{A}/\theta)$. But $\beta \sqsubseteq \alpha$ implies $\mathbf{A}/\beta \in \mathbf{HS}(\mathbf{A}/\alpha) \subseteq \mathbf{V}(\mathbf{A}/\theta)$. Finally since θ is verbal, we obtain $\theta \leq \beta$, in other words, $\beta \in I(\theta)$.

Conversely, assume that $I(\theta)$ is a downset. We wish to show that if $\mathbf{A}/\psi \in \mathbf{V}(\mathbf{A}/\theta)$, then $\theta \leq \psi$. Since every congruence is a meet of members of I , it is enough to show that

$$\beta \in I \ \& \ \mathbf{A}/\beta \in \mathbf{V}(\mathbf{A}/\theta) \implies \beta \in I(\theta).$$

Observe that $\theta = \bigwedge I(\theta)$ so $\mathbf{V}(\mathbf{A}/\theta) = \mathbf{V}\{\mathbf{A}/\alpha : \alpha \in I(\theta)\}$. Since \mathbf{A}/β is subdirectly irreducible, we deduce (by Jónsson's lemma and the fact that the variety is finitely generated and congruence-distributive) that \mathbf{A}/β lies in $\mathbf{HS}\{\mathbf{A}/\alpha : \alpha \in I(\theta)\}$. Consequently, for some $\alpha \in I(\theta)$ we have $\beta \sqsubseteq \alpha$. But by assumption, $I(\theta)$ is a downset, so $\beta \in I(\theta)$ as desired. \square

A variety is called *arithmetical* if it is both congruence-distributive and congruence-permutable. The variety is *semisimple* if every subdirectly irreducible member is simple.

Theorem 5.2. *Let \mathcal{V} be a finitely generated, semisimple, arithmetical variety. Suppose further that*

$$\mathbf{S}, \mathbf{T} \in \mathbf{V}_{\text{si}} \ \& \ \mathbf{S} \in \mathbf{HS}(\mathbf{T}) \implies \mathbf{S} \in \mathbf{IS}(\mathbf{T}). \quad (3)$$

Then every finite member of \mathcal{V} is verbose.

Proof. Let \mathbf{A} be a finite member of \mathcal{V} . Since the variety is semisimple and congruence-permutable, \mathbf{A} is isomorphic to a product $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$, in which every factor is simple (see [1, pg. 171]). Let η_i be the kernel of the projection of \mathbf{A} onto \mathbf{A}_i , for $i = 1, \dots, n$. Congruence-distributivity implies that $I^{\mathbf{A}} = \{\eta_1, \dots, \eta_n\}$.

Let θ be a congruence on \mathbf{A} and assume that θ is not verbal. Then by Lemma 5.1 $I(\theta)$ is not a downset under \sqsubseteq . Therefore there are indices i, j with $\eta_i \in I(\theta)$, $\eta_j \notin I(\theta)$ and $\eta_j \sqsubseteq \eta_i$. This latter condition, together with our assumption (3) implies that $\mathbf{A}_j \in \mathbf{S}(\mathbf{A}_i)$. Let h be an embedding of \mathbf{A}_j into \mathbf{A}_i . Summarizing

$$\theta \leq \eta_i, \quad \theta \not\leq \eta_j, \quad h: \mathbf{A}_j \hookrightarrow \mathbf{A}_i.$$

Pick $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ with $(\mathbf{a}, \mathbf{b}) \in \theta \setminus \eta_j$. Thus $a_i = b_i$ while $a_j \neq b_j$. Define $e: A \rightarrow A$ by

$$e(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, h(x_j), x_{i+1}, \dots, x_n).$$

The map e is obviously an endomorphism. But the pair $(e(\mathbf{a}), e(\mathbf{b}))$ is excluded from η_i , so certainly from θ . Thus θ is not fully invariant. \square

One assumption that implies condition (3) is *subsemisimplicity*, i.e., every nontrivial subalgebra of a subdirectly irreducible algebra is simple. It is well-known that a finitely generated, subsemisimple, arithmetical variety is a discriminator variety.

Corollary 5.3. *The finite members of a finitely generated discriminator variety are verbose.*

However, there are other varieties that satisfy the conditions of Theorem 5.2. For example, let \mathbf{A} denote the 6-element modular ortholattice \mathbf{MO}_2 (see [1, pg. 190]). The only subdirectly irreducible members of $\mathbf{V}(\mathbf{A})$ are \mathbf{A} itself and the 2-element lattice, both of which are simple, so it is easy to see that condition (3) holds. But \mathbf{A} has a 4-element subalgebra that is not simple.

The proof of Theorem 5.2 relies on the fact that every finite algebra in the variety is a direct product of simple algebras. This makes it relatively easy to construct endomorphisms with particular properties. On the other hand, the reliance on products limits the theorem to finite algebras since on cardinality grounds, it is never true that every algebra in a nontrivial variety is a direct product in a significant way.

Thus we seek varieties for which there is a representation theory that is “almost as good” as a direct product. For this we turn to the results of Davey, Keimel and Werner [5, 10] on varieties generated by a quasiprimal algebra. We follow the treatment in [4].

An algebra \mathbf{M} is *subalgebra-primal* if the clone of term operations on \mathbf{M} consists of all operations that preserve each subalgebra of \mathbf{M} . (In [4] these algebras are called semiprimal.) It is well-known that a finite algebra \mathbf{M} is subalgebra-primal if and only if

\mathbf{M} is quasiprimal and the only isomorphisms between nontrivial subalgebras are the identity maps. (4)

Let \mathbf{M} be subalgebra-primal, $\text{Sub}(\mathbf{M})$ denote the set of subuniverses of \mathbf{M} , and K the set of idempotent elements of \mathbf{M} . Thus $a \in K$ if and only if $\{a\} \in \text{Sub}(\mathbf{M})$. We define a *dual \mathbf{M} -structure* to be a quadruple $\mathbf{X} = \langle X, K^{\mathbf{X}}, S^{\mathbf{X}}, T^{\mathbf{X}} \rangle$ in which

- (a) X is a set and $T^{\mathbf{X}}$ is a Boolean topology on X ;
- (b) $K^{\mathbf{X}}$ consists of one nullary operation, $a^{\mathbf{X}}$, on X , for each $a \in K$;
- (c) $S^{\mathbf{X}}$ consists of one closed subspace, $P^{\mathbf{X}}$, for each $P \in \text{Sub}(\mathbf{M})$, such that
 - (i) $M^{\mathbf{X}} = X$;
 - (ii) $\{a\}^{\mathbf{X}} = \{a^{\mathbf{X}}\}$, for each $a \in K$;
 - (iii) $(P \cap Q)^{\mathbf{X}} = P^{\mathbf{X}} \cap Q^{\mathbf{X}}$;
 - (iv) $\emptyset^{\mathbf{X}} = \emptyset$, if \mathbf{M} has an empty subuniverse.

(We need $K^{\mathbf{X}}$ to be present to ensure that a substructure of a dual \mathbf{M} -structure is again a dual \mathbf{M} -structure.)

Let \mathcal{X} denote the class of all dual \mathbf{M} -structures. We turn \mathcal{X} into a category in which the morphisms from \mathbf{X} to \mathbf{Y} consist of all continuous functions $f: X \rightarrow Y$ such that

$$\vec{f}(P^{\mathbf{X}}) \subseteq P^{\mathbf{Y}}, \quad \text{for all } P \in \text{Sub}(\mathbf{M}).$$

Finally, let \mathcal{V} be the variety generated by \mathbf{M} . The “NU-Strong Duality Theorem” tells us that, as categories, \mathcal{V} and \mathcal{X} are dually equivalent.

To proceed, we need more details on the functors that witness this equivalence. Let \mathbf{A} be a member of \mathcal{V} . Then $D(\mathbf{A}) = \mathbf{X}$ in which

$$\begin{aligned} X &= \text{Hom}_{\mathcal{V}}(\mathbf{A}, \mathbf{M}); \\ K^{\mathbf{X}} &= \{ \mathbf{a} : a \in K \}; \\ P^{\mathbf{X}} &= \text{Hom}_{\mathcal{V}}(\mathbf{A}, \mathbf{P}), \text{ for all } P \in \text{Sub}(\mathbf{M}); \\ T^{\mathbf{X}} &\text{ is the topology on } X \text{ with subbasis } \{ T(a, m) : a \in A, m \in M \}. \end{aligned}$$

Here, $T(a, m) = \{ \phi \in X : \phi(a) = m \}$. Also, if a is an idempotent element of \mathbf{M} , then \mathbf{a} denotes the constant function on A with value a . Note that in this case, $\text{Hom}_{\mathcal{V}}(\mathbf{A}, \{a\}) = \{ \mathbf{a} \}$. Finally, for $\mathbf{A}, \mathbf{B} \in \mathcal{V}$, and a homomorphism $h: \mathbf{A} \rightarrow \mathbf{B}$, we have $D(h): D(\mathbf{B}) \rightarrow D(\mathbf{A})$ mapping ϕ to $\phi \circ h$. Then D is a contravariant functor from \mathcal{V} to \mathcal{X} .

We must also describe the functor E operating in the opposite direction. We give M the discrete topology, and turn M into a dual \mathbf{M} -structure in the obvious way. Given $\mathbf{X} \in \mathcal{X}$, define $E(\mathbf{X}) = \text{Hom}_{\mathcal{X}}(\mathbf{X}, M) \subseteq M^X$. It is not hard to show that $\text{Hom}_{\mathcal{X}}(\mathbf{X}, M)$ is, in fact, a subuniverse of \mathbf{M}^X . Consequently, $E(\mathbf{X}) \in \mathcal{V}$.

Let $\mathbf{A} \in \mathcal{V}$. Theorem 3.14 of [4] asserts that there is an isomorphism $\varepsilon_{\mathbf{A}}: \mathbf{A} \rightarrow ED(\mathbf{A})$ mapping $a \in A$ to \hat{a} , in which $\hat{a}(\tau) = \tau(a)$. We usually omit the subscript on ε . Unwrapping all of the definitions:

$$\begin{aligned} \text{Let } \mathbf{X} &= D(\mathbf{A}) = \text{Hom}_{\mathcal{V}}(\mathbf{A}, \mathbf{M}) \\ \forall a \in A, \quad \varepsilon(a) &= \hat{a} \in E(X), \text{ i.e.,} \\ \hat{a}: \text{Hom}_{\mathcal{V}}(\mathbf{A}, \mathbf{M}) &\rightarrow M \text{ such that} \\ \forall \tau \in \text{Hom}_{\mathcal{V}}(\mathbf{A}, \mathbf{M}), \quad \hat{a}(\tau) &= \tau(a). \end{aligned}$$

Let us extract from this the following relationship.

$$\begin{aligned} (\forall a, b \in A) (\forall \tau \in \text{Hom}_{\mathcal{V}}(\mathbf{A}, \mathbf{M})) \\ (a, b) \in \ker \tau \iff \tau(a) = \tau(b) \iff \hat{a}(\tau) = \hat{b}(\tau). \end{aligned} \tag{5}$$

Continuing, with $\mathbf{X} = D(\mathbf{A})$, let $f: \mathbf{X} \rightarrow \mathbf{X}$ be an \mathcal{X} -morphism. Then $\check{f} = E(f): E(\mathbf{X}) \rightarrow E(\mathbf{X})$ is given by

$$\check{f}(\hat{a}) = \hat{a} \circ f$$

see Figure 3. Finally, if we define e to be $\varepsilon^{-1} \circ \check{f} \circ \varepsilon$, then e is an endomorphism of \mathbf{A} in which, for $a \in A$, we have $e(a) = \varepsilon^{-1} \check{f} \varepsilon(a)$, and thus

$$\widehat{e(a)} = \check{f} \varepsilon(a) = \check{f}(\hat{a}) = \hat{a} \circ f. \tag{6}$$

With this dual-equivalence in hand, we can essentially imitate the construction in Theorem 5.2.

Theorem 5.4. *Every subalgebra-primal algebra generates a verbose variety.*

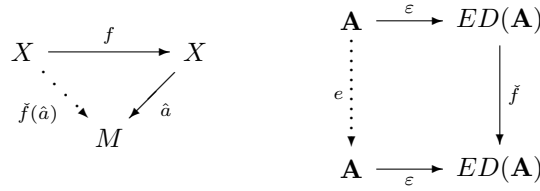


FIGURE 3

Proof. Let \mathbf{M} be subalgebra-primal and \mathcal{V} be the variety generated by \mathbf{M} . Choose an algebra \mathbf{A} in \mathcal{V} and set $\mathbf{X} = D(\mathbf{A})$ as above. Let θ be a congruence on \mathbf{A} that is not verbal. We must show that θ is not fully invariant. By Lemma 5.1 there are $\alpha, \beta \in I^{\mathbf{A}}$ such that $\theta \leq \alpha$, $\theta \not\leq \beta$ and $\beta \sqsubseteq \alpha$.

Since \mathbf{M} is subalgebra-primal, we can apply the characterization in (4). Thus the subdirectly irreducible members of \mathcal{V} coincide with the nontrivial subalgebras of \mathbf{M} , all of which are simple. From the isomorphism assertion in (4), there are unique $\phi, \psi \in X$ such that $\ker \phi = \beta$ and $\ker \psi = \alpha$. Setting $P = \overrightarrow{\phi}(A)$ and $Q = \overrightarrow{\psi}(A)$, we have that both \mathbf{P} and \mathbf{Q} are nontrivial subalgebras of \mathbf{M} and (since $\beta \sqsubseteq \alpha$), $\mathbf{P} \leq \mathbf{Q}$.

Let $Q = \{u_1, \dots, u_n\}$. Choose $c_1, \dots, c_n \in A$ such that $\psi(c_i) = u_i$, for $i \leq n$. Since each $\hat{c}_i: X \rightarrow M$ is continuous, the set

$$N = \bigcap_{i \leq n} \overleftarrow{c}_i(u_i) = \{ \tau \in X : (\forall i \leq n) \tau(c_i) = u_i \}$$

is a clopen subset of X . Note that $\psi \in N$.

We define a function $f: X \rightarrow X$ by

$$f(\tau) = \begin{cases} \phi, & \text{if } \tau \in N, \\ \tau, & \text{if } \tau \notin N. \end{cases}$$

We shall show that f is a morphism of \mathcal{X} . First, to see that f is continuous, let U be an open set. Then

$$\overleftarrow{f}(U) = \begin{cases} N \cup (U \cap N') = U \cup N, & \text{if } \phi \in U, \\ U \cap N', & \text{if } \phi \notin U \end{cases}$$

where we write N' for the complement of N . Now suppose that $R \in \text{Sub}(\mathbf{M})$. Consider any $\tau \in R^{\mathbf{X}} = \text{Hom}_{\mathcal{V}}(\mathbf{A}, \mathbf{R})$. If $\tau \notin N$ then $f(\tau) = \tau \in R^{\mathbf{X}}$. On the other hand, suppose that $\tau \in N$. Then $f(\tau) = \phi$. But $\tau \in N$ implies that $\tau(c_i) = u_i$, for $i \leq n$, so $Q \leq \overrightarrow{\tau}(A) \leq R$. Since $P \leq Q$ and $\overrightarrow{\phi}(A) = P$, we conclude $\phi = f(\tau) \in R^{\mathbf{X}}$.

Thus f is an endomorphism of \mathbf{X} , so $e = \varepsilon^{-1} \circ \check{f} \circ \varepsilon$ is an endomorphism of \mathbf{A} . Since $\theta \not\leq \beta$ we can choose $(a, b) \in \theta \setminus \beta$. Also $\alpha \in I(\theta)$ implies $\psi(a) = \psi(b)$, hence, from (5), $\hat{a}(\psi) = \hat{b}(\psi)$. By construction, $\psi \in N$, so

$f(\psi) = \phi$. Therefore, from (6),

$$\widehat{e(a)}(\psi) = \hat{a}(f(\psi)) = \hat{a}(\phi) = \phi(a)$$

and similarly, $\widehat{e(b)}(\psi) = \phi(b)$. Since $\phi(a) \neq \phi(b)$, we conclude from (5) (with ψ replacing τ and $(e(a), e(b))$ replacing (a, b)) that $(e(a), e(b)) \notin \ker \psi \geq \theta$. Hence θ is not fully invariant. \square

We can use this theorem to answer a question raised in Section 4.

Example 5.5. Let $A = \{0, 1, 2\}$ and $B = \{0, 1\}$. Consider the clone C of all operations on A that preserve the subset B . The algebra $\mathbf{A} = \langle A, C \rangle$ is subalgebra-primal and has a single proper subuniverse, namely B . It follows from Theorem 5.4 that $\mathcal{V} = \mathbf{V}(\mathbf{A})$ is verbose. And of course, \mathcal{V} is finitely generated and congruence-distributive.

However, the simple algebra \mathbf{A} is not projective in \mathcal{V} . If it were, then the first projection map $\mathbf{A} \times \mathbf{B} \rightarrow \mathbf{A}$ would have a one-sided inverse, j . In that case j followed by the second projection would yield a homomorphism from \mathbf{A} to \mathbf{B} . But no such homomorphism exists, as we see by considering the cardinality of A and B , the simplicity of \mathbf{A} , and the fact that \mathbf{B} has no idempotent elements.

6. Conclusion

Fully invariant congruence relations (also known as fully characteristic congruence relations) have long been of interest in the study of algebraic structures. Verbal congruence relations have also been of interest, especially in group theory where they appear as normal verbal subgroups. A verbal congruence relation is always fully invariant, but the converse is not true in general. In this paper we introduce the notion of a verbose algebra as an algebra in which every fully invariant congruence relation is verbal. A classic example of a verbose algebra is any free algebra in a variety. We call a variety verbose if every algebra in the variety is verbose.

This paper contains a number of results that provide sufficient conditions for an algebra or a variety to be verbose. We are very far from providing a complete characterization of verbose varieties. Several of our results call out for strengthening by relaxing their hypotheses. We present some of these as open problems.

The hypotheses of Theorem 4.2 include congruence distributivity. A natural question is whether this can be weakened to congruence modularity.

Problem 1. If \mathcal{V} is a finitely generated, congruence modular variety such that every subdirectly irreducible member of \mathcal{V} is projective, must \mathcal{V} be verbose?

The results of Theorem 2.8, Corollary 4.4, and Theorem 4.7 suggest such a strengthening of Theorem 4.2 might be possible.

There appears to be a fundamental difference between finite and infinite algebras in our investigations. For example, compare Theorem 2.2 with Example 2.5 or compare Theorem 3.1 with the Prüfer group discussion following Example 2.9. The behavior of infinite algebras is central to the next two questions.

Problem 2. Suppose \mathcal{V} is a locally finite variety in which every finite algebra is verbose. Must \mathcal{V} be verbose?

The next question asks if the finiteness condition in Theorem 4.3 can be relaxed.

Problem 3. Is a variety \mathcal{V} verbose if it contains exactly one subdirectly irreducible algebra?

Theorem 5.2 and Corollary 5.3 provide conditions on a semisimple arithmetical variety that guarantee all finite members of the variety are verbose. The more restrictive hypothesis of subalgebra-primal in Theorem 5.4 guarantees that all algebras in the variety are verbose. These results suggest the following two questions:

Problem 4. Let \mathcal{V} be any semisimple arithmetical variety for which

$$\mathbf{S}, \mathbf{T} \in \mathbf{V}_{\text{si}} \ \& \ \mathbf{S} \in \mathbf{HS}(\mathbf{T}) \implies \mathbf{S} \in \mathbf{IS}(\mathbf{T}).$$

Is \mathcal{V} verbose?

Problem 5. Is every variety generated by a quasiprimal algebra verbose?

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CLIFFORD BERGMAN

Department of Mathematics, Iowa State University, Ames, Iowa 50011 USA
e-mail: cbergman@iastate.edu

JOEL BERMAN

Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, 851 S. Morgan, Chicago, IL 60607 USA
e-mail: jberman@uic.edu