Remarks on categorical equivalence of finite unary algebras

1. Background

M. Krasner's original theorems from 1939 say that a finite algebra $\mathbf{A}$

(1) is an essentially multiunary algebra in which all operations are permutations if and only if for every n, $\mathrm{Sub}(\mathbf{A}^n)$ is closed under (set-theoretic) complementation;

(2) is an essentially multiunary algebra if and only if for every n, $\mathrm{Sub}(\mathbf{A}^n)$ is closed under unions.

By an "essentially multiunary algebra" I mean an algebra in which each basic operation is essentially unary, (i.e., depends on only one variable). From now on, I'll just refer to these as unary algebras. The algebras in the first part of Krasner's theorem will be called "group actions".

Motivated by number (1), I proved the following (see attached manuscript "Boolean Krasner Algebras").

Theorem 1. *Let* $\mathbf{A}$ *be a finite algebra with no nullary operations, and suppose that for every* n, $\mathrm{Sub}(\mathbf{A}^n)$ *is a Boolean lattice. Then* $\mathbf{A}$ *is categorically equivalent to a group action.*

The definition of categorical equivalence is from [1, Definition 0.1]. Of course the definitive treatment of categorical equivalence is McKenzie's paper [4]. Based on my extravagant success in Theorem 1, I made the following conjecture.

Conjecture 2. *Let* $\mathbf{A}$ *be a finite algebra with no nullary operations. Then* $\mathbf{A}$ *is categorically equivalent to a unary algebra if and only if, for every* n, $\mathrm{Sub}(\mathbf{A}^n)$ *is a distributive lattice.*

However, I have had no luck in proving this. Let me say at the outset that I am not at all clear on the role to be played by constant operations (either nullary operations or constant n-ary operations—which are, after all, "essentially nullary"). For one thing, my proof of Theorem 1 does not work if we drop the "no nullary operations" clause and try to prove that $\mathbf{A}$ is categorically equivalent to a "group action with constants". For another, Emil Kiss showed me that the variety of pointed rectangular bands is subalgebra-distributive (in fact, every term is sufficiently unary—see Definition 3) but is not categorically equivalent to a unary algebra. For now, I'm content to disallow all constant operations and try to get a theorem.

If $\mathbf{A}$ is a unary algebra with no constant operations then $\mathbf{V}(\mathbf{A})$ is a topos. In fact, varieties of unary algebras (a.k.a. M-sets, for some monoid M) provide one of the motivating examples of classes of topoi. Peter Johnstone in [3] characterized those varieties that are topoi. Here is how his theorem goes.

Definition 3. Let p be an n-ary term in the language of a variety $\mathcal{V}$. We say that p is *sufficiently unary in* $\mathcal{V}$ if there exists an m-ary term q and unary terms $u_1, \dots, u_m$ such that in $\mathcal{V}$:

$$u_i(p(x_1, \dots, x_n)) \text{ is essentially unary, for } i = 1, \dots, n \text{ and}$$
$$q(u_1(x), \dots, u_m(x)) \approx x$$

Theorem 4 (Johnstone)**.** *A variety $\mathcal{V}$ is a topos if and only if it has no constant operations and every term is sufficiently unary in $\mathcal{V}$.*

As I'll argue below, if every term is sufficiently unary in $\mathcal{V}$, then every algebra in $\mathcal{V}$ has a distributive subalgebra lattice. This suggests the following modification of Conjecture 2.

Conjecture 5. *Let $\mathcal{V}$ be a finitely generated (locally finite?) variety. Then $\mathcal{V}$ is a topos if and only if $\mathcal{V}$ is categorically equivalent to $\mathsf{V}(\mathbf{B})$ for some unary algebra $\mathbf{B}$ without constant operations.*

2. Problems

There are several problems we might want to consider. They are closely related.

Characterize those finite algebras $\mathbf{A}$

(1) that are categorically equivalent to a unary algebra;
(2) for which $\mathrm{Sub}(\mathbf{A}^n)$ is distributive (for every n) up to categorical equivalence;
(3) such that $\mathsf{V}(\mathbf{A})$ is a topos.

Furthermore, What about constant and/or nullary operations? Can Theorem 1 be extended to cover that case?

With regards to question 2, one should consult [5], which has a characterization of subalgebra-distributive varieties.

3. Some things we know

It follows from McKenzie's theorem that for any two algebras $\mathbf{A}$ and $\mathbf{B}$, we have $\mathbf{A} \equiv_{\mathrm{c}} \mathbf{B}$ if and only if there is an integer n and an invertible idempotent term s on $\mathbf{B}^{[n]}$ such that $\mathbf{B}^{[n]}(s) \equiv_{\mathrm{t}} \mathbf{A}$. Here '$\equiv_{\mathrm{c}}$' means categorically equivalent, and '$\equiv_{\mathrm{t}}$' means term-equivalent. However, every unary algebra is "c-minimal", see the remark after Lemma 5.2 of [2]. Therefore we get the following improvement on McKenzie.

Theorem 6. *If $\mathbf{A}$ is finite and essentially unary and if $\mathbf{B} \equiv_{\mathrm{c}} \mathbf{A}$, then there is an invertible idempotent term s on $\mathbf{B}$ such that $\mathbf{B}(s) \equiv_{\mathrm{t}} \mathbf{A}$.*

For any set A, let

$$\delta^k_{ij} = \{\, (x_1, x_2, \dots, x_k) \in A^k : x_i = x_j \,\}.$$

Also, δ is shorthand for δ^2_{12}.

Proposition 7. (1) *An algebra* $\mathbf{A}$ *is unary if and only if the ternary relation* $\delta^3_{12} \cup \delta^3_{23}$ *is a subalgebra of* $\mathbf{A}^3$.

(2) *A finite algebra* $\mathbf{A}$ *is a group action iff the relation* $A^2 - \delta$ *is a subalgebra of* $\mathbf{A}^2$.

Proofs of both of these can be found in Theorem 1.3.1 of [6]. (But the real work is in Theorem 1.1.6.) This suggests two more questions:

(4) Is it true that a finite algebra $\mathbf{B}$ is categorically equivalent to a unary algebra iff $\mathrm{Sub}(\mathbf{B}^3)$ is distributive?
(5) Are either of these conditions sufficient to conclude that $\mathbf{B}$ is categorically equivalent to a group action: $\mathrm{Sub}(\mathbf{B}^2)$ is Boolean or δ has a complement in $\mathrm{Sub}(\mathbf{B}^2)$?

4. Things true about unary algebras

Let $\mathbf{A}$ be a finite unary algebra with no constant operations, and let $\mathcal{V}$ be the variety generated by $\mathbf{A}$. The following list contains some things that are true about $\mathbf{A}$ and $\mathcal{V}$. A few of them require a bit of explanation.

An algebra $\mathbf{B}$ is *Hamiltonian* if every subalgebra is a class of some congruence on $\mathbf{B}$. A variety is Hamiltonian just in case each member is Hamiltonian.

The coproduct of two unary algebras (of the same similarity type and containing no constant operations) is their disjoint union. Suppose that $\mathbf{B}$ is a subalgebra of the unary algebra $\mathbf{A}$. Then there is a congruence $\theta(B)$ on $\mathbf{A} \uplus \mathbf{A}$ which identifies the two copies of each element of B (and nothing else).

Theorem 8. *Let* $\mathbf{A}$ *be a unary algebra with no constant operations and let* $\mathcal{V} = \mathbf{V}(\mathbf{A})$. *All of the following are true.*

(1) *For every* n, $\mathrm{Sub}(\mathbf{A}^n)$ *is distributive.*
(2) *For every* $\mathbf{B}, \mathbf{C} \in \mathcal{V}$, *every homomorphism* $f\colon \mathbf{B} \to \mathbf{C}$, *and every* $X, Y \in \mathrm{Sub}(\mathbf{C})$, $f^{-1}(X \vee Y) = f^{-1}(X) \vee f^{-1}(Y)$. *(Join is in the lattice of subuniverses.)*
(3) *Every term of* $\mathcal{V}$ *is sufficiently unary.*
(4) *For every* $\alpha \in \mathrm{Con}(\mathbf{A} \amalg \mathbf{A})$ *such that* $\alpha \subseteq \theta(A)$, *there is a subalgebra* $\mathbf{B}$ *of* $\mathbf{A}$ *such that* $\alpha = \theta(B)$.
(5) *Every member of* $\mathcal{V}$ *is Hamiltonian.*
(6) *Every epimorphism in* $\mathcal{V}$ *is surjective.*
(7) $\mathcal{V}$ *is strongly Abelian.*
(8) $\mathcal{V}$ *has the congruence extension property.*
(9) $\mathcal{V}$ *is a topos.*

I'm fairly sure that each of these properties is preserved by categorical equivalence. Condition 4 is due to Keith Kearnes, who pointed out that the congruence $\theta(B)$ is the kernel (in the universal algebraic sense) of the difference cokernel of the two canonical inclusion maps of $\mathbf{B}$ into $\mathbf{A} \amalg \mathbf{A}$.

Here is what I know about the relationships between these conditions.

(9) $\iff$ (3) This is Theorem 4.

(3) $\iff$ (2) This requires a proof. Assume (3) holds. Let $f\colon \mathbf{B} \to \mathbf{C}$ be a homomorphism, $X, Y \in \mathrm{Sub}(\mathbf{C})$. Since inverse images of functions always preserves set-theoretic inclusion, $f^{-1}(X) \vee f^{-1}(Y) \subseteq f^{-1}(X \vee Y)$. For the opposite inclusion, suppose that $b \in f^{-1}(X \vee Y)$. Thus $f(b) \in X \vee Y$, and therefore there are $a \in X$, $c \in Y$ and $g \in \mathrm{Clo}_2(\mathbf{C})$ such that $f(b) = g(a, c)$. Now by assumption, g is sufficiently unary. Therefore there are terms $u_1, \dots, u_m$ and q satisfying the conditions in Definition 3. So (computing in $\mathbf{B}$), $b = q(u_1(b), \dots, u_m(b))$. But, since f is a homomorphism, for all $i \le m$, $f(u_i(b)) = u_i(f(b)) = u_i(g(a, c)) \in X \cup Y$ since $u_i \circ g$ is essentially unary. Therefore $u_i(b) \in f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y) \subseteq f^{-1}(X) \vee f^{-1}(Y)$, hence $b = q(u_1(b), \dots, u_m(b)) \in f^{-1}(X) \vee f^{-1}(Y)$.

For the converse, we follow Jonstone's argument. Suppose that (2) holds. We use $\mathbf{F}_{\mathcal{V}}(X)$ to denote the $\mathcal{V}$-free algebra over the set X. Let p be an n-ary term. We wish to prove that p is sufficiently unary. Then there is a homomorphism f from $\mathbf{F}_{\mathcal{V}}(y)$ to $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x_1, \dots, x_n)$ mapping y to p. Let X_i be the subalgebra of $\mathbf{F}$ generated by the single element x_i. Since $x_1, \dots, x_n$ generate $\mathbf{F}$, $X_1 \vee X_2 \vee \cdots \vee X_n = F$. By assumption $F_{\mathcal{V}}(y) = f^{-1}(F) = f^{-1}(X_1 \vee \cdots \vee X_n) = \bigvee_{i=1}^{n} f^{-1}(X_i)$. Since $y \in F(y)$, there is an n-ary term q and unary terms $u_1, \dots, u_n$ such that $u_i(y) \in f^{-1}(X_i)$ for $i = 1, \dots, n$ and $y = q(u_1(y), \dots, u_n(y))$. This gives one of the conditions from Definition 3. To get the other, $u_i(p(x_1, \dots, x_n)) = u_i(f(y)) = f(u_i(y)) \in X_i = \mathrm{Sg}^{\mathbf{F}}(x_i)$, hence, there is a unary term v_i such that $u_i(p(x_1, \dots, x_n)) = v_i(x_i)$.

(2) $\implies$ (1) Let X, Y, Z be subalgebras of $\mathbf{A}$. Let $f\colon \mathbf{X} \hookrightarrow \mathbf{A}$ be the inclusion map. Then $f^{-1}(Y \vee Z) = X \cap (Y \vee Z)$ while $f^{-1}(Y) \vee f^{-1}(Z) = (X \cap Y) \vee (X \cap Z)$.

(1) $\implies$ (7)+(5)+(8) This follows from Palfy [5], Theorems 6 and 9.

5. A few miscellaneous remarks

When he presents the definition of sufficiently unary, Johnstone specifically says that $m \neq n$ in general (m and n as in Definition 3). But it seems to me that it follows from my proof of (2) implies (3) above that m and n will be equal. Have I missed something?

One of Johnstone's conditions for a topos is that "coproducts are universal". For a variety, is this precisely equivalent to condition (2)?

In light of Theorem 6, given a finite algebra $\mathbf{A}$ satisfying some of the conditions in the list above, we must find an invertible idempotent term s such that $\mathbf{A}(s)$ is unary. It would be useful to have a condition on s that is equivalent to $A(s)$ being unary. One idea is to somehow utilize Proposition 7. Another tack might be via Theorems 2.6–2.8 in Johnstone's paper. (See also, Proposition 9 below.) Note that if $M = \mathrm{Clo}_1(\mathbf{A})$ is the monoid of unary terms of $\mathbf{A}$, then the corresponding monoid of $\mathbf{A}(s)$ is sMs. When does this latter monoid have the left Ore condition mentioned in Johnstone's paper?

One more question comes to mind. Do either of conditions (4) or (6) from the list in Section 4 hold in every topos?

6. An observation on minimal idempotents

Let $\mathbf{A}$ be an algebra, and let $\mathrm{ii}(\mathbf{A})$ denote the set of invertible idempotent terms on $\mathbf{A}$. Consider the following true facts.

(1) Let $\mathbf{A}$ be an algebra, $s, t \in \mathrm{ii}(\mathbf{A})$ and assume that $t(A) \subseteq s(A)$. Then $t_s \in \mathrm{ii}(\mathbf{A}(s))$ and $\mathbf{A}(t) \equiv_{\mathrm{t}} \mathbf{A}(s)(t_s)$.
(2) Let $s \in \mathrm{ii}(\mathbf{A})$ and t a term on $\mathbf{A}$ such that $t_s \in \mathrm{ii}(\mathbf{A}(s))$. Then $sts \in \mathrm{ii}(\mathbf{A})$ and $\mathbf{A}(sts) \equiv_{\mathrm{t}} \mathbf{A}(s)(t_s)$.
(3) Every unary algebra is c-minimal.
(4) On a unary algebra, the only invertible idempotent term operation is the identity.

Of these statements, (1) and (2) are easy to verify (and (2) is essentially remark 3 following Theorem 2.3 of [4]). (3) is Theorem 6, and (4) is trivial.

We can quasi-order the members of $\mathrm{ii}(\mathbf{A})$ by $t \lesssim s$ iff $t(A) \subseteq s(A)$. Call an invertible idempotent term s *minimal* if for all $t \in \mathrm{ii}(\mathbf{A})$, $t \lesssim s \implies t(A) = s(A)$.

Proposition 9. *Suppose $\mathbf{A}$ is a finite algebra that is categorically equivalent to a unary algebra. Let $s \in \mathrm{ii}(A)$. Then $\mathbf{A}(s)$ is unary iff s is minimal.*

Proof. First suppose that $\mathbf{A}(s)$ is unary and that $t \lesssim s$. By (1) t_s is an invertible idempotent term of $\mathbf{A}(s)$, so by (4) t_s is the identity on $s(A)$. But then, for all $a \in s(A)$, $a = st(a) = t(a)$ (since $t(A) \subseteq s(A) = ss(A)$), so $s(A) \subseteq t(A)$. Thus s is minimal.

Conversely, assume that s is minimal in $\mathrm{ii}(\mathbf{A})$. Since $\mathbf{A}(s) \equiv_{\mathrm{c}} \mathbf{A}$, $\mathbf{A}(s)$ is categorically equivalent to a unary algebra. By (3), there is a term t on $\mathbf{A}$ such that $t_s \in \mathrm{ii}(\mathbf{A}(s))$ and $\mathbf{A}(s)(t_s)$ is unary. But by (2) $sts \in \mathrm{ii}(\mathbf{A})$ and $\mathbf{A}(sts) \equiv_{\mathrm{t}} \mathbf{A}(s)(t_s)$ is unary. Obviously, $sts \lesssim s$, so by the minimality of s, $sts(A) = s(A)$. Therefore t_s is a permutation of $s(A)$. Since t_s is also idempotent, it must be the identity. Thus $\mathbf{A}(s)$ is a unary algebra. □

This proposition (along with c-minimality) tells us that if we wish to prove that a finite algebra $\mathbf{A}$ is categorically equivalent to a unary algebra, we should be able to take any minimal invertible idempotent term s (and no others) and prove that $\mathbf{A}(s)$ is unary.

So, suppose we have some magic condition M (which is preserved by categorical equivalence) and we wish to prove that any finite algebra satisfying M is categorically equivalent to a unary algebra. We might proceed as follows. Let $\mathbf{A}$ be a finite algebra satisfying M and assume that $\mathbf{A}$ has no invertible idempotent terms (except the identity), and try to prove that $\mathbf{A}$ must be unary. How are we going to prove that $\mathbf{A}$ is unary? Well one way would be to use Proposition 7(1).

For example, suppose that condition M includes the assumption that every term of $\mathbf{A}$ is sufficiently unary. Now if $\mathbf{A}$ is not unary, then $\delta_{12} \cup \delta_{23}$

is not a subuniverse of $\mathbf{A}^3$. Therefore, there are elements $a, b, c, d \in A$ and a binary term p such that (computing in $\mathbf{A}^3$)

$$p\big((a, a, b),\ (c, d, d)\big) \notin \delta_{12} \cup \delta_{23}.$$

In other words (computing in $\mathbf{A}$)

$$p(a, c) \neq p(a, d) \neq p(b, d). \tag{1}$$

By assumption p is sufficiently unary. Therefore, there is a binary term q and unary terms u_1, u_2, v_1, v_2 such that

$$q(u_1(x),\ u_2(x)) = x \tag{2}$$

$$u_1(p(x, y)) = v_1(x), \quad u_2(p(x, y)) = v_2(y) \tag{3}$$

are identities that hold in $\mathbf{A}$.

I'm trying to find a way to contradict the assumption that $\mathbf{A}$ has no nontrivial invertible idempotents. Combining lines (1) and (3) we deduce that neither u_1 nor u_2 are the identity.(In fact they are not permutation.) But they are presumably not idempotent. Maybe there is some way to manipulate them and use equation (2) to obtain a nontrivial invertible idempotent term. But so far, I don't see it. [I tried an example the other day. I wound up with terms satisfying $u_1 u_2 = u_2$ and $u_2 u_1 = u_1$. So I obtained the identity $q(u_1(x), u_1 u_2(x)) = x$ which is enough to prove that u_1 is invertible.]

Well, I don't know where to go with these ideas. Maybe use one of the other conditions from Theorem 8.

References

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