SUBQUASIVARIETIES OF REGULARIZED VARIETIES

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Abstract. This paper considers the lattice of subquasivarieties of a regular variety. In particular we show that if V is a strongly irregular variety that is minimal as a quasivariety, then the smallest quasivariety containing both V and Sl (the variety of semilattices) is never equal to the regularization $\widetilde{\mathsf{V}}$ of V.

We use this result to describe the lattice of subquasivarieties of $\widetilde{\mathsf{V}}$ in several special, but quite common, cases and give a number of applications and examples.

1. Introduction

Let V be an irregular variety of plural type. (These terms are defined in Section 2.) The *regularization* $\widetilde{\mathsf{V}}$ of V is the smallest variety containing both V and Sl. Here, Sl denotes the variety of the same type as V and term-equivalent to the variety of semilattices. The assignment $\mathsf{V} \mapsto \widetilde{\mathsf{V}}$ acts as an operator that picks out precisely the regular identities valid in V. There is a nice theory, going back to J. Płonka, describing the structure of the variety $\widetilde{\mathsf{V}}$ in the case that V is *strongly irregular,* i.e., V satisfies an identity of the form $x*y = x$ for some binary term $x*y$. In particular, this identity alone axiomatizes V relative to $\widetilde{\mathsf{V}}$, see [Me] or [R].

This paper initiates a study of the lattice of quasivarieties, $\mathcal{L}_{\mathcal{Q}}(\widetilde{\mathsf{V}})$, of a regularized variety. More precisely, we are interested in the relationship between $\mathcal{L}_{\mathcal{Q}}(\widetilde{\mathsf{V}})$ and $\mathcal{L}_{\mathcal{Q}}(\mathsf{V})$. In this context, it is natural to consider an analog of the construction of $\widetilde{\mathsf{V}}$ from V, namely, the smallest quasivariety containing both V and Sl. We call this quasivariety the *quasi-regularization* of V.

After recalling the necessary facts concerning regularized varieties and Płonka sums in Section 2, we show in Section 3 that the quasi-regularization of a strongly irregular variety V can be defined, relative to $\widetilde{\mathsf{V}}$, by a single quasi-identity. The variety $\widetilde{\mathsf{V}}$ is known to consist precisely of Płonka sums of V-algebras. We show that the members of the quasi-regularization of V are exactly those Płonka sums in which every Płonka homomorphism is injective.

In [DG] it is shown that for V strongly irregular, the lattice of subvarieties of $\widetilde{\mathsf{V}}$, $\mathcal{L}_{\mathcal{V}}(\widetilde{\mathsf{V}})$, is closely related to $\mathcal{L}_{\mathcal{V}}(\mathsf{V})$. In fact, $\mathcal{L}_{\mathcal{V}}(\widetilde{\mathsf{V}}) \cong \mathcal{L}_{\mathcal{V}}(\mathsf{V}) \times \mathcal{L}_{\mathcal{V}}(\mathsf{Sl})$. It seems reasonable to hope that an analogous result might hold for the lattice of quasivarieties. We explore this question in the simplest possible case: for a locally finite, strongly irregular variety V which is minimal as a quasivariety and such that every member of V has an idempotent element. For such a variety we prove in Theorem 4.3 that the lattice of subquasivarieties of the quasi-regularization of V is isomorphic to $\mathcal{L}_{\mathcal{Q}}(\mathsf{V}) \times \mathcal{L}_{\mathcal{Q}}(\mathsf{Sl})$. (By our assumptions on V, this lattice is nothing

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but the Cartesian square of the two-element chain.) In Sections 5–6 we extend this result to a characterization of $\mathcal{L}_{\mathcal{Q}}(\widetilde{\mathsf{V}})$ under some additional conditions: first, that every member of V is an idempotent algebra, and second, that V has a Mal'cev term (or, more generally, V is weakly congruence regular). In both cases, we show that the only subquasivariety that is not a variety is the quasi-regularization. Both of these sections contain a large number of examples and applications. In particular, we provide an example (constant semigroups) to show that strong irregularity is essential in the results just described.

Finally, in Section 7 we consider a counterpart to the results in Section 6. Let V be a minimal Mal'cev variety in which no member has an idempotent element. V is always generated by a quasiprimal algebra. We prove that, in contrast to the 5 element lattice obtained in the idempotent case, $\mathcal{L}_{\mathcal{Q}}(\widetilde{\mathsf{V}})$ is a nine element lattice, and the lattice of subquasivarieties of the quasi-regularization of V has 5 elements.

The notation and terminology of the paper is basically as in the book [RS2]. We refer the reader to [BS], [MMT] and [RS2] for undefined notions and results. We use "Polish" notation for functions and operations, e.g., $\omega(x_1, x_2, \ldots, x_n)$ denotes a term with variables x_1, x_2, $\ldots$,x_n, and then $x_1x_2 \ldots x_n\omega$ denotes the induced term operation on an algebra. An Ω-algebra with universe A will frequently be abbreviated to $\mathbf{A}$ when we do not need to explicitly indicate the type. We write $\mathbf{A} \leq \mathbf{B}$ to indicate that the algebra $\mathbf{A}$ is isomorphic to a subalgebra of $\mathbf{B}$.

2. Regularized varieties and Płonka sums

Let Ω be a fixed set of operation symbols. A type of algebras is called *plural* if it contains at least one fundamental operation symbol of arity greater than 1 and no nullary operation symbols. It is easy to see that the class of all algebras of a fixed plural type $\rho\colon \Omega \to \mathbb{N}$ contains a subvariety that is term-equivalent to the variety of semilattices by defining, for each semilattice $\langle S, \cdot\rangle$ and each n-ary operation symbol ω of Ω,

$$x_1x_2 \ldots x_n\omega := x_1 \cdot x_2 \cdots x_n.$$

Notice that every unary operation symbol will be interpreted as the identity map under this definition. The semilattice operation can be retrieved via the definition $x \cdot y := xyy \ldots y\omega$, for any operation ω of arity greater than 1. This subvariety shall be denoted Sl_Ω, or simply Sl, if no confusion will result. Throughout this paper, we will work in the fixed plural type ρ.

An identity is called *regular* if the same variables appear on each side of the equal sign. A variety is called regular if it satisfies only regular identities. Observe that Sl_Ω is the smallest regular variety of type ρ, since its identities are precisely all regular identities. At the opposite extreme, a variety is *strongly irregular* if it satisfies an identity of the form

$$x * y = x \tag{2–1}$$

for some binary term $x{*}y$. It is well-known that such a variety has a basis consisting of regular identities together with the single identity $x * y = x$. See [Me], [R], [PR], or [RS2]. Since Sl satisfies no irregular identities, we have $\mathsf{V} \cap \mathsf{Sl} = \mathsf{1}$ (the variety of trivial algebras) for any irregular variety V. It seems fair to say that most interesting varieties are strongly irregular. For example, every congruence modular variety has this attribute. Any congruence modular variety has a set of Day terms. So in particular, there will be a 4-ary term τ such that $\mathsf{V} \models xyyx\tau = x$, and we can take $x * y = xyyx\tau$.

The *regularization* of a variety V, denoted $\widetilde{\mathsf{V}}$, is the variety defined by all regular identities that hold in V. Equivalently, $\widetilde{\mathsf{V}} = \mathbf{V}(\mathsf{V}, \mathsf{Sl}_\Omega)$, the smallest variety of Ω-algebras containing both V and Sl_Ω. A beautiful structure theory for the members of $\widetilde{\mathsf{V}}$ has been developed in the case that V is strongly irregular. We summarize it here.

First recall that a semilattice $\langle S, \cdot \rangle$ may be regarded as an idempotent, commutative semigroup; as an ordered set $\langle S, \leq_\cdot \rangle$, where $x \leq_\cdot y$ iff $x \cdot y = x$; and also as a (small) category (S) with object-set S and such that for each $s, t \in S$, $\hom(s,t) = \{s \rightarrow t\}$ if $s \leq_\cdot t$ and $\hom(s,t) = \varnothing$ otherwise. Furthermore, as described above, a semilattice may be considered to be an Ω-algebra $\langle S, \Omega \rangle$.

Let F be a contravariant functor from the category (S) to the variety V, viewed as a category with all of its homomorphisms as the morphisms of the category. For each $s \in S$ let $\mathbf{A}_s = sF$, and for each $s \leq_\cdot t$ in S let $\varphi_{t,s} = (s \rightarrow t)F$. Then the *Płonka sum of* F, or more verbosely, the Płonka sum of the system $\langle \mathbf{A}_s : s \in S;\ \varphi_{t,s} : s \leq_\cdot t \rangle$ is the Ω-algebra $\mathbf{A}$ with universe $A = SF = \biguplus (A_s : s \in S)$, and with, for each n-ary $\omega \in \Omega$, a basic operation given by

$$\text{(2–2)} \qquad \begin{aligned} &\omega^{\mathbf{A}} \colon A_{s_1} \times A_{s_2} \times \cdots \times A_{s_n} \rightarrow A; \\ &\quad (x_1, x_2, \ldots, x_n) \mapsto x_1\varphi_{s_1,s} x_2 \varphi_{s_2,s} \cdots x_n \varphi_{s_n,s} \omega^{\mathbf{A}_s} \end{aligned}$$

where $s_1, s_2, \ldots, s_n \in S$ and $s = s_1 \cdot s_2 \cdots s_n$. The *canonical projection* of this Płonka sum is the homomorphism $\pi_F \colon \mathbf{A} \rightarrow \langle S, \Omega \rangle$ given by $A_s \pi_F = \{s\}$. The semilattice $\langle S, \Omega \rangle$ is called the *semilattice replica* of the algebra $\mathbf{A}$ ([Ma], [RS2]). The kernel of π_F, which we usually denote by σ is the semilattice replica congruence.

It is easy to see that each algebra $\mathbf{A}_s$ is a subalgebra of $\mathbf{A}$. These subalgebras are referred to as the Płonka fibers, and the maps $\varphi_{t,s}$ as the Płonka homomorphisms or as the fiber maps of $\mathbf{A}$.

With this preparation, we can state the following basic structure theorem.

Theorem 2.1. ([P1], [P3], [PR], [R], [RS2]) *Let V be a strongly irregular variety of Ω-algebras defined by a set Σ of regular identities and a single identity of the form $x * y = x$. Then the following classes coincide.*

1. *The regularization, $\widetilde{\mathsf{V}}$ of V;*
2. *The class $\mathbf{P\ell}(\mathsf{V})$ of Płonka sums of V-algebras;*
3. *The variety of Ω-algebras defined by the identities Σ and the following identities (for $\omega \in \Omega$, with $n = \omega\rho$):*

$$\text{(2–3)} \qquad \begin{aligned} x * x &= x, \\ (x * y) * z &= x * (y * z), \\ x * y * z &= x * z * y, \\ x_1 \ldots x_n \omega * y &= (x_1 * y) \ldots (x_n * y)\omega, \\ y * (x_1 \ldots x_n \omega) &= y * x_1 * \cdots * x_n. \end{aligned}$$

In the variety V, the identities (2–3) are consequences of $x * y = x$. In $\widetilde{\mathsf{V}}$, the operation $x * y$ is called a partition operation, since it serves to decompose $\widetilde{\mathsf{V}}$-algebras into Płonka sums of V-algebras. In fact the semilattice replica congruence σ, can be obtained on each $\widetilde{\mathsf{V}}$-algebra $\mathbf{A}$ by

$$a \, \sigma \, b \iff a * b = a \;\; \& \;\; b * a = b. \tag{2–4}$$

Similarly, for the fiber map $\varphi_{t,s} \colon A_t \to A_s$ we have $x\varphi_{t,s} = x * y$, where y is an arbitrary member of A_s. If the semilattice S and at least one of the fibers A_s are non-trivial, then the Płonka sum will be called *non-trivial* as well.

For an investigation into the lattice $\mathcal{L}_{\mathcal{Q}}(\widetilde{\mathsf{V}})$, it would obviously be helpful to have information about the lattice $\mathcal{L}_{\mathcal{V}}(\widetilde{\mathsf{V}})$ of subvarieties of $\widetilde{\mathsf{V}}$, and of the subdirectly irreducible algebras in $\widetilde{\mathsf{V}}$. This is contained in Theorems 2.3 and 2.4 below.

Definition 2.2. Let $\mathbf{A}$ be any algebra and $S_2 = \langle \{0,1\}, \leq_. \rangle$ be a two-element semilattice (with $0 <_. 1$). The algebra $\mathbf{A}^\infty$ is the Płonka sum of the functor F, where $F(1) = \mathbf{A}$ and $F(0)$ is the trivial algebra $1 = \langle \{\infty\}, \Omega \rangle$ of the same type as $\mathbf{A}$.

Note that the algebra 1^∞ is, up to isomorphism, the unique two-element member of Sl_Ω.

Theorem 2.3. [LPP] *Let V be a strongly irregular variety. The subdirectly irreducible members of $\widetilde{\mathsf{V}}$ are the algebras $\mathbf{A}$ and $\mathbf{A}^\infty$, as $\mathbf{A}$ ranges over all subdirectly irreducible algebras of V, and the algebra 1^∞, where 1 denotes a trivial V-algebra.*

Theorem 2.4. [DG] *Let V be a strongly irregular variety. The lattice $\mathcal{L}_{\mathcal{V}}(\widetilde{\mathsf{V}})$ of subvarieties of $\widetilde{\mathsf{V}}$ is isomorphic to the lattice $\mathcal{L}_{\mathcal{V}}(\mathsf{V}) \times \mathcal{L}_2$, where $\mathcal{L}_2$ denotes the two-element lattice.*

It is worth noting that if a basis for the identities of V is given as in Theorem 2.1, then each subvariety W of V is defined by the basis for V, together with some additional regular identities. According to Theorem 2.1, a basis for $\widetilde{\mathsf{W}}$ can be obtained from that of $\widetilde{\mathsf{V}}$ by the addition of those same regular identities.

What of the case that V is irregular, but not strongly irregular? We can still form Płonka sums of V-algebras, and the variety $\widetilde{\mathsf{V}}$ still exists and still contains $\mathbf{Pł}(\mathsf{V})$. However, in this case, $\widetilde{\mathsf{V}}$ does not necessarily consist precisely of Płonka sums of V-algebras, and the Płonka sums lose many of the nice properties that they have in the strongly irregular case. In particular, the Płonka homomorphisms between the fibers are not uniquely determined, and Theorem 2.1 does not hold, in general.

3. The quasi-regularization of an irregular variety

Let V be an irregular variety, and let $\mathbf{A} \in \widetilde{\mathsf{V}}$. Assume that $\mathbf{A}$ is a Płonka sum of its subalgebras $(\mathbf{A}_s : s \in S)$ via the functor $F \colon (S) \to \mathsf{V}$. Recall that σ denotes the semilattice replica congruence on $\mathbf{A}$, and that $\mathbf{A}/\sigma \cong (S, \Omega)$.

Define two other binary relations on the set A. For $s, t \in S$, a_s in A_s and b_t in A_t let

$$\begin{aligned} a_s \, \delta \, b_t &\iff \exists u \in S.\;\; a_s\varphi_{s,u} = b_t\varphi_{t,u}; \\ a_s \, \gamma \, b_t &\iff t \leq_. s \;\; \& \;\; b_t = a_s\varphi_{s,t}. \end{aligned} \tag{3–1}$$

For any binary relation α and β, $\alpha^\smile$ denotes the converse of α, and $\alpha \circ \beta$ denotes the relative product of α and β.

Lemma 3.1. *Let* V *be an irregular variety and* $\mathbf{A}$ *a Płonka sum of a functor* $F\colon (S) \to \mathsf{V}$*. Then*

(1) δ *is a congruence relation on* $\mathbf{A}$*;*
(2) $\mathbf{A}/\delta$ *is the colimit of the functor* F*;*
(3) $\mathbf{A}/\delta \in \mathsf{V}$*;*
(4) $\delta = \gamma \circ \gamma^{\smile}$*;*
(5) *In* $\operatorname{Con}\mathbf{A}$, $\delta \vee \sigma = 1_A$*, the largest congruence on* $\mathbf{A}$.

Proof. Statements (1)–(3) are easy consequences of well-known facts (see [G, §21] and [P1]). Let us consider (4). Obviously, $\gamma \subseteq \delta$, and therefore, since δ is an equivalence relation, $\gamma \circ \gamma^{\smile} \subseteq \delta$. Now let $a_s \ \delta \ b_t$, for $s, t \in S$, $a_s \in A_s$ and $b_t \in A_t$. Then there is $u \in S$ with $c_u := a_s\varphi_{s,u} = b_t\varphi_{t,u}$. Note that $a_s \ \delta \ c_u \ \delta \ b_t$. On the other hand, by the definition of γ, one has $a_s \ \gamma \ c_u \ \gamma^{\smile} \ b_t$. From this we conclude that $\delta \subseteq \gamma \circ \gamma^{\smile}$, so (4) holds.

Finally, for (5), we note that the algebra $\mathbf{A}/(\delta \vee \sigma)$ is a homomorphic image of both $\mathbf{A}/\delta$ and $\mathbf{A}/\sigma$. Consequently, $\mathbf{A}/(\delta \vee \sigma) \in \mathsf{V} \cap \mathsf{Sl}_\Omega$. But this latter variety consists only of trivial algebras. □

For an arbitrary irregular variety V, the congruence δ defined above is dependent on the choice of the functor F. If V is strongly irregular, then each algebra $\mathbf{A}$ in $\widetilde{\mathsf{V}}$ determines a unique functor F (with codomain V). Therefore, (3–1) defines a unique congruence on each member of $\widetilde{\mathsf{V}}$.

Let $x * y$ denote a binary term of Ω. We let q_* denote the following quasi-identity of Ω.

$$(q_*) \quad (x * y = x \ \ \& \ \ y * x = y \ \ \& \ \ x * z = z * x = z \ \ \& \ \ y * z = z * y = z) \ \rightarrow \ (x = y)$$

Lemma 3.2. *Let* V *be a strongly irregular variety satisfying the identity* $x * y = x$*. Then*

(1) $\mathsf{V} \models q_*$*;*
(2) $\mathsf{Sl}_\Omega \models q_*$*;*
(3) *For any nontrivial* V*-algebra* $\mathbf{A}$, $\mathbf{A}^\infty \not\models q_*$.

Proof. In V, the antecedent of q_* implies that $x = x * z = z * x = z$ and similarly $y = z$, from which we obtain $x = y$. And in Sl_Ω, we have $x = x * y = y * x = y$.

Now let a and b be distinct elements of a V-algebra $\mathbf{A}$. Let $\infty = a\varphi_{1,0} = b\varphi_{1,0}$ in $\mathbf{A}^\infty$. Note that, by (2–2), $a * \infty = (a\varphi_{1,0}) * (\infty\varphi_{0,0}) = \infty * \infty = \infty$. Using this and similar computations, we see that, with a, b, ∞ as witnesses for x, y, z; q_* fails to hold in $\mathbf{A}^\infty$. □

As usual, for a class K of similar algebras, let $\mathbf{P}(\mathsf{K})$ denote the class of algebras isomorphic to a product of members of K, and $\mathbf{S}(\mathsf{K})$ the class of algebras isomorphic to subalgebras of K-algebras. Also, $\mathbf{Q}(\mathsf{K})$ denotes the quasivariety generated by K.

As a consequence of the above Lemma, we have the following (sharp) inclusions, for a strongly irregular variety V.

$$\mathsf{V},\ \mathsf{Sl}_\Omega \subset \mathbf{Q}(\mathsf{V} \cup \mathsf{Sl}_\Omega) \subset \widetilde{\mathsf{V}}. \tag{3–2}$$

Proposition 3.3. *Let* $\mathbf{A}$ *be an algebra in the regularization of a strongly irregular variety* V*. Assume that* $\mathbf{A}$ *is the Płonka sum of its subalgebras* $(\mathbf{A}_s : s \in S)$ *over the semilattice* S*, with Płonka homomorphisms* $\varphi_{s,t}$*. The following are equivalent.*

(1) $\mathbf{A} \in \mathsf{SP}(\mathsf{V} \cup \mathsf{Sl}_\Omega)$;
(2) *For every* $s \geq t$ *in* S*, the homomorphism* $\varphi_{s,t}$ *is injective;*
(3) *In* $\operatorname{Con}\mathbf{A}$*,* $\delta \wedge \sigma = 0_A$*, the identity relation on* A*.*
(4) $\mathbf{A} \models q_*$.

Proof. (1)⇒(2): Without loss of generality, we can assume that $\mathbf{A}$ is a subalgebra of a product $\mathbf{B} \times \mathbf{T}$, in which $\mathbf{B} \in \mathsf{V}$ and $\mathbf{T} \in \mathsf{Sl}_\Omega$. The algebra $\mathbf{B} \times \mathbf{T}$ is a Płonka sum of subalgebras $\mathbf{B}_t$ for $t \in T$ over the semilattice T, where $B_t = B \times \{t\}$. The Płonka homomorphisms on $\mathbf{B}$ are all isomorphisms. Now the Płonka homomorphisms on a subalgebra are the restrictions of those on the larger algebra. Therefore the Płonka homomorphisms on $\mathbf{A}$ are simply the restrictions of those on $\mathbf{B} \times \mathbf{T}$, and are therefore injective.

(2)⇒(3): Suppose $a \equiv b \pmod{\delta \wedge \sigma}$. Then there are $s, u \in S$ with $s \geq u$ such that $a, b \in A_s$ and $a\varphi_{s,u} = b\varphi_{s,u}$. Since $\varphi_{s,u}$ is injective, we get $a = b$.

(3)⇒(1): Since $\delta \wedge \sigma = 0$, we have a subdirect embedding of $\mathbf{A}$ into $\mathbf{A}/\delta \times \mathbf{A}/\sigma$. By Lemma 3.1(3), $\mathbf{A}/\delta \in \mathsf{V}$, and $\mathbf{A}/\sigma$ is isomorphic to the Ω-semilattice $\mathbf{S}$. Thus $\mathbf{A} \in \mathsf{SP}(\mathsf{V} \cup \mathsf{Sl}_\Omega)$.

(4)⇒(2): Let $\mathbf{A}$ satisfy q_*; let $s \geq t$ in S; $a, b \in A_s$ and suppose $a\varphi_{s,t} = b\varphi_{s,t}$. Substituting $a, b, a\varphi_{s,t}$ for x, y, z in q_*, we conclude that $a = b$.

(1)⇒(4): By Lemma 3.2 every member of $\mathsf{V} \cup \mathsf{Sl}_\Omega$ satisfies q_*. Since the satisfaction of quasi-identities is inherited by both subalgebras and products,

$$\mathsf{SP}(\mathsf{V} \cup \mathsf{Sl}_\Omega) \models q_*. \quad \square$$

Let us note that the assumption that V be *strongly* irregular is needed only in Proposition 3.3(4). Moreover, the condition in 3.3(1) can obviously be replaced by

(1′) $\mathbf{A}$ is in the class $\mathsf{P_s}(\mathsf{V}_{\mathrm{si}} \cup \{\mathbf{S}_2\})$,

where V_{si} is the class of subdirectly irreducible V-algebras, $\mathbf{S}_2$ is the two-element Ω-semilattice, and $\mathsf{P_s}(\mathsf{K})$ denotes the class of all algebras isomorphic to subdirect products of members of K.

Definition 3.4. Let V be an irregular variety of type Ω. The *quasi-regularization* of V is the quasivariety $\mathsf{Q}(\mathsf{V}, \mathsf{Sl}_\Omega)$, i.e., the smallest quasivariety containing both V and Sl_Ω. If $x * y$ is a binary Ω-term, then we define $\widetilde{\mathsf{V}}_{q_*}$ to be the subquasivariety of $\widetilde{\mathsf{V}}$ defined by the quasi-identity q_*.

In the strongly irregular case, we put all of this together in the following Theorem.

Theorem 3.5. *Let* V *be a strongly irregular variety satisfying* $x * y = x$*. Then the quasi-regularization of* V *is equal to* $\widetilde{\mathsf{V}}_{q_*}$*. Specifically,*

$$\widetilde{\mathsf{V}}_{q_*} = \mathsf{SP}(\mathsf{V} \cup \mathsf{Sl}_\Omega) = \mathsf{Q}(\mathsf{V}, \mathsf{Sl}_\Omega) = \mathsf{P_s}(\mathsf{V}_{\mathrm{si}} \cup \{\mathbf{S}_2\}).$$

Proof. Lemma 3.2 and Proposition 3.3 imply the inclusions:

$$\mathsf{Q}(\mathsf{V}, \mathsf{Sl}_\Omega) \subseteq \widetilde{\mathsf{V}}_{q_*} = \mathsf{SP}(\mathsf{V} \cup \mathsf{Sl}_\Omega) \subseteq \mathsf{Q}(\mathsf{V}, \mathsf{Sl}_\Omega). \quad \square$$

In light of Theorem 3.5, we see that $\widetilde{\mathsf{V}}_{q_*}$ is independent of the choice of the term $x * y$ (so long as $\mathsf{V} \models x * y = x$). In the sequel, we write $\widetilde{\mathsf{V}}_q$ instead of $\widetilde{\mathsf{V}}_{q_*}$.

Let M denote the variety of Ω-algebras defined by the single identity $x * y = x$. M is a maximal strongly irregular variety. Furthermore, every subvariety of M is defined (relative to M) by a set of regular identities. The regularization $\widetilde{\mathsf{M}}$ of M is defined by the identities in (2–3). More generally, by Theorem 2.4, there is a lattice isomorphism $\mathsf{V} \mapsto \widetilde{\mathsf{V}}$ between the subvarieties of M and the class of regular subvarieties of $\widetilde{\mathsf{M}}$. Let us refer to this latter class as $\mathcal{L}^r_{\mathcal{V}}$. This class can also be thought of as the interval sublattice $[\,\mathsf{Sl}_\Omega, \widetilde{\mathsf{M}}\,]$ of $\mathcal{L}_{\mathcal{V}}(\widetilde{\mathsf{M}})$.

Similarly, let $\mathcal{L}^r_{\mathcal{Q}}$ denote the class $\{\, \widetilde{\mathsf{V}}_q : \mathsf{V} \in \mathcal{L}_{\mathcal{V}}(\mathsf{M}) \,\}$, ordered by inclusion. We have the following relationship.

Theorem 3.6. *The poset $\mathcal{L}^r_{\mathcal{Q}}$ is a lattice. The three lattices $\mathcal{L}_{\mathcal{V}}(\mathsf{M})$, $\mathcal{L}^r_{\mathcal{Q}}$ and $\mathcal{L}^r_{\mathcal{V}}$ are isomorphic via the mapping* $\mathsf{V} \mapsto \widetilde{\mathsf{V}}_q \mapsto \widetilde{\mathsf{V}}$.

Proof. As we mentioned above, the isomorphism between $\mathcal{L}_{\mathcal{V}}(\mathsf{M})$ and $\mathcal{L}^r_{\mathcal{V}}$ follows from Theorem 2.4. And the mapping from $\mathcal{L}_{\mathcal{V}}(\mathsf{M})$ to $\mathcal{L}^r_{\mathcal{Q}}$ is certainly surjective and order-preserving. For injectivity it suffices to prove that if $\mathsf{V} \subseteq \mathsf{M}$ then $\mathsf{V} = \mathsf{M} \cap \widetilde{\mathsf{V}}_q$. Recall that by strong irregularity, V is defined relative to $\widetilde{\mathsf{V}}$ by the identity $x*y = x$. Therefore

$$\mathsf{V} = \mathsf{M} \cap \widetilde{\mathsf{V}} \supseteq \mathsf{M} \cap \widetilde{\mathsf{V}}_q \supseteq \mathsf{V}. \quad \square$$

Theorem 3.6 says nothing about the subquasivarieties of M that fail to be varieties. In particular, we do not know how such a quasivariety will join with Sl_Ω. Furthermore, we know very little about the quasivarieties that lie between a variety and its regularization.

We conclude this section with the following open problems.

Problem 3.7. Which quasi-identities satisfied in a subquasivariety N of M are inherited by the subquasivariety $\mathbf{Q}(\mathsf{N}, \mathsf{Sl}_\Omega)$ of $\widetilde{\mathsf{M}}$? Is there a set Σ of axioms for N relative to M such that $\mathbf{Q}(\mathsf{N}, \mathsf{Sl}_\Omega)$ is axiomatized by Σ relative to $\widetilde{\mathsf{M}}_q$? Is $\mathsf{N} = \mathbf{Q}(\mathsf{N}, \mathsf{Sl}_\Omega) \cap \mathsf{M}$?

Problem 3.8. Which subquasivarieties of $\widetilde{\mathsf{M}}_q$ are of the form $\mathbf{Q}(\mathsf{N}, \mathsf{Sl}_\Omega)$ for some $\mathsf{N} \in \mathcal{L}_{\mathcal{Q}}(\mathsf{M})$?

Problem 3.9. Develop a theory of quasi-regular quasivarieties. Find both syntactical (see 3.7) and semantical (Płonka sums with injective homomorphisms?) characterizations of the quasi-regularization of a quasivariety.

4. Varieties minimal as quasivarieties

Definition 4.1. Let K be a class of algebras. An algebra is called K-*prime* if it can be embedded into every non-trivial member of K.

A variety is called *equationally complete* if it contains no proper, non-trivial subvarieties, in other words, it is a minimal (non-trivial) variety. In the next few sections, we consider varieties that are not only minimal as a variety, but also minimal as a quasivariety. Some sufficient conditions for this to hold were given in [BM]. In particular, we have the following Theorem.

Theorem 4.2. [BM] *Let* V *be a variety. If* V *has, up to isomorphism, a unique subdirectly irreducible algebra* $\mathbf{A}$*, and if* $\mathbf{A}$ *is* V*-prime then* V *is a minimal quasivariety. Conversely, any locally finite variety which is minimal as a quasivariety has such an algebra* $\mathbf{A}$*.*

The variety Sl_Ω satisfies the conditions of Theorem 4.2. The unique subdirectly irreducible member is the two-element Ω-semilattice $\mathbf{S}_2$, which embeds into each non-trivial member of the variety. Thus Sl_Ω is a minimal quasivariety.

In the sequel, we denote the variety generated by a single algebra $\mathbf{A}$ by $\mathsf{V}(\mathbf{A})$ and the quasivariety generated by $\mathbf{A}$ by $\mathsf{Q}(\mathbf{A})$. We use a similar convention for other classes generated by $\mathbf{A}$: $\mathsf{P}(\mathbf{A})$, $\mathsf{P_s}(\mathbf{A})$, etc. We extend this convention to classes generated by a finite set of algebras as well. Thus $\mathsf{V}(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n)$ denotes the variety generated by $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$. We remind the reader that if $\mathbf{A}_1, \dots, \mathbf{A}_n$ are finite, then $\mathsf{Q}(\mathbf{A}_1, \dots, \mathbf{A}_n) = \mathsf{SP}(\mathbf{A}_1, \dots, \mathbf{A}_n)$, see [BS], Lemma IV.6.5 and Theorem V.2.25.

An element e of an algebra $\mathbf{A}$ is called an *idempotent element* if $\{e\}$ is a subuniverse of $\mathbf{A}$. The algebra $\mathbf{A}$ is called idempotent if every member is an idempotent element, and a variety is called idempotent if every member algebra is idempotent.

Theorem 4.3. *Let* V *be a locally finite, strongly irregular variety of* Ω*-algebras, each of which contains an idempotent element. If* V *is minimal as a quasivariety, then the lattice* $\mathcal{L}_\mathcal{Q}(\widetilde{\mathsf{V}}_q)$ *of subquasivarieties of the quasiregularization of* V *consists of the following four members:*

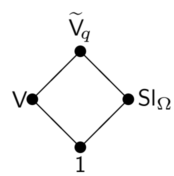

Figure 1. $\mathcal{L}_\mathcal{Q}(\widetilde{\mathsf{V}}_q)$

Proof. By Theorem 4.2, V has a unique finite subdirectly irreducible algebra $\mathbf{A}$ which is V-prime, and $\mathsf{V} = \mathsf{SP}(\mathbf{A})$. Let Z be a non-trivial subquasivariety of $\widetilde{\mathsf{V}}_q$ distinct from both V and Sl. Since both V and Sl are minimal quasivarieties, there is an algebra $\mathbf{B} \in \mathsf{Z} - (\mathsf{V} \cup \mathsf{Sl})$. Then $\mathbf{B}$ is a non-trivial Płonka sum of V-algebras $\langle \mathbf{B}_s : s \in S \rangle$ over the semilattice S.

The non-triviality of the Płonka sum has two consequences. First, there is $u \in S$ such that $\mathbf{B}_u$ is a non-trivial V-algebra. Therefore, $\mathbf{A}$ embeds into $\mathbf{B}_u$, hence into $\mathbf{B}$. It follows that

$$\mathsf{V} = \mathsf{SP}(\mathbf{A}) \subseteq \mathsf{Z}. \tag{4–1}$$

The second consequence is that $|S| > 1$. Therefore, there are $s >\cdot t$ in S. By assumption, the algebra $\mathbf{B}_s$ has an idempotent element, b_s. It is then easy to see that $\{b_s, b_s\varphi_{s,t}\}$ forms the universe of a subalgebra of $\mathbf{B}$ isomorphic to $\mathbf{S}_2$. This implies that

$$\mathsf{Sl}_\Omega = \mathsf{SP}(\mathbf{S}_2) \subseteq \mathsf{Z}. \tag{4–2}$$

Combining (4–1) and (4–2) we obtain

$$\widetilde{\mathsf{V}}_q = \mathbf{Q}(\mathsf{V}, \mathsf{Sl}_\Omega) \subseteq \mathsf{Z} \subseteq \widetilde{\mathsf{V}}_q. \quad \square$$

As we shall see in Section 7, the existence of an idempotent is essential in this Theorem.

5. Some applications to idempotent algebras

Theorem 4.3 describes the lattice $\mathcal{L}_\mathcal{Q}(\widetilde{\mathsf{V}}_q)$ under certain conditions. Of course it would be much more interesting to have a description of $\mathcal{L}_\mathcal{Q}(\widetilde{\mathsf{V}})$. In the next three sections we give such a description under several different sets of hypotheses.

Theorem 5.1. *Let* V *be a locally finite, idempotent, strongly irregular variety of* Ω*-algebras that is minimal as a quasivariety. Then the lattice of subquasivarieties of the regularized variety* $\widetilde{\mathsf{V}}$ *consists of the following five members:*

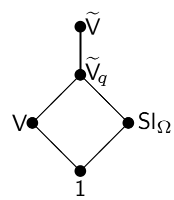

Figure 2. $\mathcal{L}_\mathcal{Q}(\widetilde{\mathsf{V}})$

Proof. By Theorem 4.3 we need only show that if Z is a quasivariety such that $\mathsf{Z} \subseteq \widetilde{\mathsf{V}}$ and $\mathsf{Z} \nsubseteq \widetilde{\mathsf{V}}_q$, then $\mathsf{Z} = \widetilde{\mathsf{V}}$. Let $\mathbf{A}$ be the unique subdirectly irreducible member of V. By Theorem 2.3, $\widetilde{\mathsf{V}}_{\text{si}} = \{\mathbf{A}, \mathbf{A}^\infty, \mathbf{S}_2\}$ (up to isomorphism). We always have $\mathbf{A} \leq \mathbf{A}^\infty$ (that is, $\mathbf{A}$ embeds into $\mathbf{A}^\infty$) and because of our idempotence assumption, $\mathbf{S}_2 \leq \mathbf{A}^\infty$ as well. It follows that $\widetilde{\mathsf{V}} = \mathbf{Q}(\mathbf{A}^\infty)$. To complete the proof it suffices to show that $\mathbf{A}^\infty \in \mathsf{Z}$.

Let $\mathbf{B} \in \mathsf{Z} - \widetilde{\mathsf{V}}_q$. Consider $\mathbf{B}$ as a Płonka sum of $\langle \mathbf{B}_s : s \in S \rangle$ over the semilattice S. By Proposition 3.3, there are $s >_\cdot t$ in S such that the Płonka homomorphism $\varphi_{s,t}$ is not injective. Let $a, b \in B_s$ with $c := a\varphi_{s,t} = b\varphi_{s,t}$. By assumption, c is an idempotent element of $\mathbf{B}_t$. Therefore $c\varphi_{s,t}^{-1}$ is the universe of a subalgebra $\mathbf{B}'$ of $\mathbf{B}_s$. Since $a, b \in B'$, it is a non-trivial V-algebra. Therefore, since V is a minimal quasivariety, $\mathbf{A} \leq \mathbf{B}'$. It is now easy to see that $B' \cup \{c\}$ forms a subuniverse of $\mathbf{B}$ containing a copy of $\mathbf{A}^\infty$. Thus $\mathbf{A}^\infty \in \mathsf{Z}$ as desired. $\square$

Example 5.2. The variety Dl of distributive lattices is strongly irregular: take $x * y := x \wedge (x \vee y)$. Thus it satisfies the hypotheses of Theorem 5.1. The regularization, $\widetilde{\mathsf{Dl}}$ of Dl is the variety of distributive bisemilattices; algebras with two semilattice operations satisfying all axioms of distributive lattices except for the absorption laws. (See [P2], [PR].) The variety $\widetilde{\mathsf{Dl}}$ consists precisely of Płonka sums of distributive lattices. The quasiregularization, $\widetilde{\mathsf{Dl}}_q$ of Dl is the class of all Płonka sums of distributive lattices with injective Płonka homomorphisms. Instead of the identity (q_*), one can take

$$(x \leq_\wedge y \leq_\wedge z) \;\; \& \;\; (y \leq_\vee z \leq_\vee x) \; \rightarrow \; y = z$$

as a basis for $\widetilde{\mathsf{Dl}}_q$ relative to $\widetilde{\mathsf{Dl}}$. There is a renewed interest in distributive bisemilattices both because they are the first example of dualizable Płonka sums [GR], [RS4], and because they have found applications in the theory of program semantics [L], [Pu].

Example 5.3. A *mode* $\langle A, \Omega\rangle$ is an algebra that is idempotent and entropic, i.e., each operation of Ω is a homomorphism from a direct power of the algebra to itself. Equivalently, for each $\omega, \tau \in \Omega$, the following identity holds:

$$x_{11} \dots x_{1n}\omega\, x_{21} \dots x_{2n}\omega \dots x_{m1} \dots x_{mn}\omega\tau = x_{11} \dots x_{m1}\tau \dots x_{1n} \dots x_{mn}\tau\omega$$

(where ω is n-ary and τ is m-ary). Entropic algebras have appeared under other names in the literature such as "commutative", "Abelian" and "medial". Modes were defined and studied by Romanowska and Smith in [RS2]. Examples of modes are furnished by affine spaces (or modules) and their reducts, semilattices and convex sets.

As was recently shown by Kearnes [K], every minimal variety of modes is term-equivalent to either the variety of sets, the variety of semilattices, or a variety of affine spaces. Among the varieties of modes equivalent to sets are the variety Lz of left-zero bands (semigroups satisfying $x \cdot y = x$), and the variety Rz of right-zero bands (satisfying $x \cdot y = y$). Both varieties satisfy the assumptions of Theorems 4.2 and 5.1. Hence Theorem 5.1 provides a description of the lattices of subquasivarieties of $\widetilde{\mathsf{Lz}}$, and of $\widetilde{\mathsf{Rz}}$ which are the varieties of *left-normal* and *right-normal bands,* respectively.

The variety of semilattices is, of course, regular, so we do not obtain anything new by forming its regularization. So we turn our attention to varieties of affine spaces. We first recall the definition. Let R be a commutative ring with identity, and let $E_R = \langle E, +, R\rangle$ be a module over R. For each r in R, define a binary operation

$$\underline{r}\colon E \times E \to E; \quad (x, y) \mapsto x(1 - r) + yr$$

and the Mal'cev operation

$$P\colon E \times E \times E \to E; \quad (x, y, z) \mapsto x - y + z.$$

The algebra $\langle E, \underline{R}, P\rangle$ with the set $\underline{R} = \{\, \underline{r} : r \in R\,\}$ is term-equivalent to the full idempotent reduct of the module $\langle E, +, R\rangle$. Consequently, it can be identified with the affine space (or module) over the ring R. (See [Cs], [OS], [RS1], [RS2], [Sz1].) Carrying out this identification, we refer to the algebra $\langle E, \underline{R}, P\rangle$ as an affine R-space. For a given ring R, the class of affine R-spaces forms a variety denoted by R. This variety was axiomatized in [RS2, 255].

The varieties R are strongly irregular, take $x * y = xy\underline{0}$. Among them, the minimal varieties are the varieties of affine spaces over fields. If F is a field, then the affine space $F^a := \langle F, \underline{F}, P\rangle$ is the unique subdirectly irreducible member of the variety F of affine F-spaces, it embeds into each non-trivial member of F, and $\mathsf{F} = \mathbf{SP}(F^a)$. It follows that F is minimal as a quasivariety. The proof of Theorem 5.1 goes through for F, even though it fails to be locally finite—all that is required is the unique prime, subdirectly irreducible algebra. Therefore, the lattice $\mathcal{L}_{\mathcal{Q}}(\widetilde{\mathsf{F}})$ is given by Figure 2.

Example 5.4. Many well-known varieties of groupoid modes are term-equivalent to varieties of affine modules. (See a short summary in [PRR].) So the conclusion of Theorem 5.1 holds for the minimal varieties of these groupoids. Among them are the minimal varieties of commutative binary modes (see [RS2], [RS3]). In [JK] these were called commutative idempotent and Abelian (cia-) groupoids.) These are equivalent to the varieties of affine $\mathbb{Z}_p$-spaces, for a prime integer p. Each of these is also equivalent to another interesting variety of groupoid modes, namely, a minimal variety of symmetric binary modes, defined by the identity $(x \cdot y) \cdot y = x$. (See [RS2], [Ro].)

Example 5.5. Quasigroups, considered as algebras $\langle A, \cdot, /, \backslash \rangle$, with the operations being multiplication, left division and right division respectively, form a variety. Since both of the division operations are uniquely determined by the multiplication, when presenting a quasigroup, it suffices to describe its multiplication table. Quasigroup modes are idempotent and entropic quasigroups. As was shown in [CM], the minimal varieties of quasigroup modes are precisely the varieties generated by the quasigroups $\langle F, \mathbf{f} \rangle$, where F is a finite field and $f \in F - \{0, 1\}$ generates F. Since quasigroup modes are Mal'cev modes, these varieties are equivalent to varieties of affine spaces over finite fields. It follows that Theorem 5.1 applies to the minimal varieties of quasigroup modes.

The regularization of the variety of affine F-spaces contains the important subclass of modes of affine subspaces of affine F-spaces. Let R be a commutative ring with identity. For an affine R-space $\mathbf{E} = \langle E, \underline{R}, P \rangle$, consider the set $\mathbf{ES}$ of non-empty subalgebras of $\mathbf{E}$. The set $\mathbf{ES}$ forms an algebra under the complex operations:

$$\underline{r} \colon \mathbf{ES} \times \mathbf{ES} \to \mathbf{ES}; \quad (X, Y) \mapsto \{\, xy\underline{r} : x \in X,\ y \in Y \,\}$$

for $r \in R$ and

$$P \colon \mathbf{ES} \times \mathbf{ES} \times \mathbf{ES} \to \mathbf{ES}; \quad (X, Y, Z) \mapsto \{\, xyzP : x \in X,\ y \in Y,\ z \in Z \,\}.$$

It turns out that the algebra $\langle \mathbf{ES}, \underline{R}, P \rangle$ is again a mode inheriting many of the algebraic properties of $\mathbf{E}$, see [RS2]. The following can be easily deduced from results of [PRS].

Theorem 5.6. *Let V be a variety of Ω-algebras equivalent to a variety of affine F-spaces over a field F. Let $\langle A, \Omega \rangle$ be in V and $\Omega \subseteq (\underline{F} - \{\underline{0}, \underline{1}\}) \cup \{P\}$. Then the algebra $\big\langle \langle A, \Omega \rangle \mathbf{S},\, \Omega \big\rangle$ is a Płonka sum of algebras equivalent to affine F-spaces over the semilattice $\langle A_F \mathbf{S}, + \rangle$ of the subspaces of the vector space A_F.*

For U in (the semilattice) $A_F\mathbf{S}$, the corresponding Płonka fiber A_U is the quotient $A/U = \{\, x + U : x \in A \,\}$. The ordering relation on the semilattice $A_F\mathbf{S}$ is defined by: $W \le U$ if and only if U is a subspace of W. The Płonka homomorphisms are $\varphi_{U,W} \colon A/U \to A/W; \quad x + U \mapsto x + W$.

By Theorem 5.6, it is evident that the algebras $\big\langle \langle A, \Omega \rangle \mathbf{S},\, \Omega \big\rangle$ are in the regularization $\widetilde{\mathsf{V}}$. However, they are never in the quasiregularization $\widetilde{\mathsf{V}}_q$. Indeed, for any proper subspace U of W, if $x - y \in W - U$, then $x + U \neq y + U$ but $(x + U)\varphi_{U,W} = (y + U)\varphi_{U,W}$.

6. Minimal Mal'cev Varieties with idempotents

In this section we prove, by a slight variation in the proof, an analog of Theorem 5.1 for some other minimal varieties. An algebra $\mathbf{A}$ is said to be *weakly congruence regular at* $a \in A$ if every congruence on $\mathbf{A}$ is determined by the congruence class of a. More precisely: for every $\alpha, \beta \in \operatorname{Con} \mathbf{A}$ if $a/\alpha = a/\beta$ then $\alpha = \beta$. (Here, a/α denotes the congruence class of a modulo α.) $\mathbf{A}$ is called *congruence regular* if, for every $a \in A$, it is weakly congruence regular at a. A variety is called (weakly) congruence regular if every member algebra is (weakly) congruence regular (at some point).

The appearance of the word "regular" in a second context is unfortunate. However both usages seem to be well entrenched in the literature.

There is a nice discussion of the various forms of congruence regularity in [DMS]. In particular, every weakly congruence regular variety is congruence modular, and furthermore, by [DMS, Theorem 2.2], in a weakly congruence regular variety, any algebra with an idempotent element is weakly congruence regular at an idempotent.

Theorem 6.1. *Let* V *be a locally finite, minimal, weakly congruence regular variety, and suppose that every member of* V *has an idempotent element. Then* V *is strongly irregular. The lattice* $\mathcal{L}_{\mathcal{Q}}(\widetilde{\mathsf{V}})$ *of subquasivarieties of* $\widetilde{\mathsf{V}}$ *is given by Figure 2.*

Proof. Weak congruence regularity implies congruence modularity which in turn implies strong irregularity. Since V is locally finite, minimal and congruence modular, V is a minimal quasivariety, by the main result of [BM]. Thus, by Theorem 4.2, V has a V-prime algebra $\mathbf{A}$, and $\mathbf{A}$ is the unique subdirectly irreducible algebra of V.

To determine $\mathcal{L}_{\mathcal{Q}}(\widetilde{\mathsf{V}})$, we proceed much as in the proof of Theorem 5.1. We need only show that there is no quasivariety Z such that $\widetilde{\mathsf{V}}_q \subset \mathsf{Z} \subset \widetilde{\mathsf{V}}$. Once again, we do this by showing that the subdirectly irreducible algebra $\mathbf{A}^\infty$ generating $\widetilde{\mathsf{V}}$ embeds into some member of Z.

So let $\mathbf{B}$ and $\varphi_{s,t}$ be as in the proof of 5.1. By our remarks above, $\mathbf{B}_s$ is weakly congruence regular at an idempotent element e. Let $\alpha = \ker \varphi_{s,t} \in \operatorname{Con}(\mathbf{B}_s)$. Since $\varphi_{s,t}$ is not injective, $\alpha \neq 0_{B_s}$. Therefore, by weak congruence regularity, $|e/\alpha| > 1$. Since e is an idempotent, $B' = e/\alpha$ is a non-trivial subalgebra of $\mathbf{B}_s$. But $e' = e\varphi_{s,t}$ is an idempotent element of $\mathbf{B}_t$. It follows that $B' \cup \{e'\}$ is a subuniverse of $\mathbf{B}$. Since $\mathbf{A}$ is V-prime, $\mathbf{A}$ embeds into $\mathbf{B}'$ and therefore $\mathbf{A}^\infty$ embeds into $\mathbf{B}' \cup \{e'\} \in \mathsf{Z}$. □

A variety V is called *Mal'cev* if there is a ternary term P such that

$$\mathsf{V} \models xyyP = x = yyxP.$$

Mal'cev varieties are among the best-understood in universal algebra. The special case of Mal'cev modes was discussed in Section 5. A variety is Mal'cev if and only if it is congruence permutable. In particular, every Mal'cev variety is congruence modular.

Corollary 6.2. *Let* V *be a locally finite, minimal, Mal'cev variety. Assume every member of* V *has an idempotent element. Then* $\mathcal{L}_{\mathcal{Q}}(\widetilde{\mathsf{V}})$ *is given in Figure 2.*

Proof. By [CK, Corollary 4.6] and [Q, Lemma 2.6] every locally finite, minimal Mal'cev variety is congruence regular, and therefore weakly congruence regular. The conclusion now follows from Theorem 6.1. □

In the remainder of this section we give several applications of the above Theorem and Corollary. In Section 7 we discuss the situation when the idempotence assumption is dropped from Corollary 6.2. First we would like to say a few words about the structure of the varieties considered in this section.

Let V be a locally finite, minimal, weakly congruence regular variety. By local finiteness, V contains a finite, non-trivial algebra $\mathbf{A}$ of smallest cardinality. This algebra is *strictly simple,* that is, it is simple and has no proper, non-trivial subalgebras. By the minimality of V, $\mathbf{A}$ generates V. As we pointed out above, weak congruence regularity implies congruence modularity. It follows from some standard results in commutator theory, see [FM, Theorem 12.1], that V is either Mal'cev or congruence distributive (or both); $\mathbf{A}$ is the unique subdirectly irreducible algebra in V; and that $\mathbf{A}$ is V-prime. Notice that the existence of idempotent elements in V is entirely determined by $\mathbf{A}$. If $\mathbf{A}$ has an idempotent, then by its primeness, every algebra in V has an idempotent. If $\mathbf{A}$ has no idempotent, then since it is the unique subdirectly irreducible algebra, no non-trivial member of V has an idempotent.

Suppose that V is Mal'cev. Then the algebra $\mathbf{A}$ is either Abelian or quasiprimal [FM, 12.1]. In the Abelian case, $\mathbf{A}$ must have an idempotent [FM, 12.4], and therefore every member of V has an idempotent element, so Corollary 6.2 applies. Examples 6.4 and 6.5 illustrate this case. If $\mathbf{A}$ is quasiprimal, then it may or may not have an idempotent. If it does have an idempotent, then again, we are covered by Corollary 6.2. See Example 6.5. On the other hand, if $\mathbf{A}$ has no idempotent, then the situation is quite different. This case is analyzed in Theorem 7.1.

Finally, if V is not Mal'cev, then it must be congruence distributive. If $\mathbf{A}$ has an idempotent, then Theorem 6.1 describes $\mathcal{L}_{\mathcal{Q}}(\widetilde{\mathsf{V}})$. See Example 6.6. That leaves the possibility that $\mathbf{A}$ has no idempotent elements. We have provided no examples since we do not know of any. And furthermore, we have been unable to determine what $\mathcal{L}_{\mathcal{Q}}(\widetilde{\mathsf{V}})$ would look like if such an example does exist. So we pose the following problem.

Problem 6.3. Does there exist a finite simple algebra $\mathbf{A}$ with no proper subalgebras, generating a variety that is congruence distributive and weakly congruence regular, but not Mal'cev? If $\mathbf{A}$ is such an algebra, determine the lattice of subquasivarieties of the regularization of $\mathbf{V}(\mathbf{A})$.

Note that it follows from results of A. Szendrei, see [Sz3], that an algebra $\mathbf{A}$ satisfying the above requirements must have a fundamental operation that is not surjective.

Example 6.4. Let F be a finite field. The variety of all F-vector spaces satisfies the hypotheses of Corollary 6.2. In this case the prime algebra is any 1-dimensional space, which is Abelian.

In particular, for any prime integer p, the variety A_p of Abelian groups of exponent p is term-equivalent to the variety of $\mathbb{Z}_p$-vector spaces. Therefore, $\mathcal{L}_{\mathcal{Q}}(\widetilde{\mathsf{A}_p})$ is given by Figure 2.

Example 6.5. As shown by Tarski [T], the minimal varieties of rings are exactly the varieties $\mathsf{R}_p = \mathbf{V}(\mathbb{Z}_p)$ generated by the finite fields $\mathbb{Z}_p$ of prime order p, and the varieties of zero-rings of prime characteristic. By a ring in this example, we mean an algebra $\langle R, +, -, \cdot\rangle$. In Example 7.12 we discuss the situation when we add a constant operation for the multiplicative identity element.

The varieties of zero-rings are term-equivalent to their Abelian group reducts, so the lattice of subquasivarieties is described in the previous example. The rings $\mathbb{Z}_p$ are quasiprimal. Since the unit element is not given by an operation, 0 is an idempotent element. Therefore, Corollary 6.2 applies and yields a description of $\mathcal{L}_{\mathcal{Q}}(\widetilde{\mathsf{R}}_p)$.

Example 6.6. An *implication algebra* is an algebra $\langle A, \rightarrow\rangle$ satisfying the following three axioms:

$$(x \rightarrow y) \rightarrow x = x$$
$$(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$$
$$x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z).$$

The variety of implication algebras satisfies the hypotheses of Theorem 6.1, but is not Mal'cev. The variety is well-known to be congruence distributive. Every algebra is weakly congruence regular at the point $x \rightarrow x$ for any x, (the equation $x \rightarrow x = y \rightarrow y$ can be derived from the above axioms) and this point is an idempotent. The unique subdirectly irreducible algebra is the reduct of the two-element Boolean algebra to the term $x \rightarrow y := x' \vee y$. Figure 2 gives the lattice of subquasivarieties of the regularization. The regularization of the variety of implication algebras was studied by Kalman [Ka] under the name quasi-implication algebras.

It is tempting to try to modify this example to answer Problem 6.3. If we begin with the two-element implication algebra $\mathbf{A} = \langle\{0,1\}, \rightarrow\rangle$ defined above and add a constant unary term with value 0, we do indeed get a finite simple algebra with no subalgebras etc. However, $\mathbf{A}$ is term equivalent to the two-element Boolean algebra as can be seen by defining $x' := x \rightarrow 0$ and $x \vee y := (x \rightarrow y) \rightarrow y$. Thus the generated variety will be Mal'cev.

A number of other minimal varieties that serve as example of applications of Theorems 5.1 and 6.1 can be found in the excellent survey of A. Szendrei [Sz2]. Our last example in this section examines the minimal varieties of semigroups. It is interesting to note that every such variety, save one, can be handled by either Theorem 5.1 or Theorem 6.1. The one anomalous variety behaves somewhat differently from the others, and requires special treatment.

Example 6.7. The minimal varieties of semigroups were described by Kalicki and Scott in [KS]. They are the varieties S_p generated by the group $\mathbb{Z}_p$ for p a prime; the varieties of left- and right-zero bands, Lz and Rz; the variety of semilattices Sl; and the variety of constant semigroups, C, defined by the identity $x \cdot y = z \cdot w$.

The variety S_p is term-equivalent to A_p discussed in Example 6.4. The varieties Lz, Rz and Sl were considered in Example 5.3. So we are left with the variety of constant semigroups. This variety contains a unique subdirectly irreducible algebra, $\mathbf{C}_2$, the two-element constant semigroup. C is therefore a finitely generated, hence locally finite variety and furthermore, it is obvious that $\mathbf{C}_2$ is C-prime. Therefore, by Theorem 4.2, C is a minimal quasivariety.

Although C is irregular, it is not strongly irregular, so Theorem 2.1 does not apply per se. However, results of [Mi] show that the regularization $\widetilde{\mathsf{C}}$ consists exactly of

Płonka sums of constant semigroups, although the Płonka homomorphisms are not uniquely defined. In fact, for $\mathbf{A} \in \widetilde{\mathsf{C}}$ with fibers A_s over the semilattice replica S, and for any $a_s, b_s, c_s, d_s \in A_s$ and $e_t, f_t, g_t, h_t \in A_t$ one has $(a_s b_s)\varphi_{s,t} = (c_s d_s)\varphi_{s,t} = e_t f_t = g_t h_t$. All other elements of A_s may be mapped into A_t in an arbitrary way.

We wish to determine the subdirectly irreducible members of $\widetilde{\mathsf{C}}$. (Since C is not strongly irregular, Theorem 2.3 does not apply.) Let $\mathbf{A} \in \widetilde{\mathsf{C}}_{\text{si}}$. Let us denote by z_s the constant element of A_s, for $s \in S$. For any $a_s \in A_s$ and $b_t \in A_t$ we have

$$a_s \cdot b_t = z_{st}$$

where st denotes the meet in S of s and t. Let θ be a congruence on S and define

$$\bar{\theta} := \{ (z_s, z_t) : (s,t) \in \theta \} \cup \{ (a,a) : a \in A \}. \tag{6–1}$$

One easily checks that $\bar{\theta}$ is a congruence on $\mathbf{A}$ and that

$$\overline{\bigcap_{\theta \in \Theta} \theta} = \bigcap_{\theta \in \Theta} \bar{\theta}$$

for any set $\Theta \subseteq \operatorname{Con} S$. Then the subdirect irreducibility of $\mathbf{A}$ implies that S is simple or trivial, in other words, $|S| \leq 2$.

If S is trivial, then $\mathbf{A} \in \mathsf{C}$, so $\mathbf{A} \cong \mathbf{C}_2$. So suppose that $S = \{0, 1\}$ with $0 < 1$. Then the argument used for Statement 2 of [LPP] still works in the present situation to show that A_0 must be trivial. Let σ denote the semilattice replica congruence (so that $z_1 \equiv x \not\equiv z_0 \pmod{\sigma}$ for any $x \in A_1$), and let $\alpha = \overline{1_S}$ (defined in (6–1), so that $z_0 \equiv z_1 \not\equiv x \pmod{\alpha}$ for any $x \in A_1 - \{z_1\}$). Then $\sigma \cap \alpha = 0_A$. From the subdirect irreducibility of $\mathbf{A}$ we conclude that $\alpha = 0_A$, and therefore $\mathbf{A} \cong \mathbf{S}_2$.

We have shown that every subdirectly irreducible algebra in $\widetilde{\mathsf{C}}$ is isomorphic to either $\mathbf{S}_2$ or to $\mathbf{C}_2$. Thus

$$\widetilde{\mathsf{C}} = \mathbf{P_s}(\mathbf{C}_2, \mathbf{S}_2) = \mathbf{Q}(\mathsf{C}, \mathsf{SI}),$$

in other words, the regularization and the quasiregularization of C are equal.

Let us note that the variety $\widetilde{\mathsf{C}}$ is *deductive* as defined in [B], i.e., each subquasivariety of $\widetilde{\mathsf{C}}$ is a variety. By Lemma 3.2, and especially inclusions (3–2), we see that the regularization of a strongly irregular variety is never deductive. This suggests the following problem.

Problem 6.8. Let V be a deductive, irregular (but not strongly so) variety. Under what conditions is $\widetilde{\mathsf{V}}$ deductive? In particular, are the regularization and quasiregularization of V identical?

Problem 6.9. Let V be a deductive, strongly irregular variety. What additional subquasivarieties will $\widetilde{\mathsf{V}}$ have besides those of the form $\widetilde{\mathsf{W}}_q$ for some subvariety W of V?

Some examples of irregular, deductive varieties are: any proper subvariety of the variety of Abelian groups [B]; the variety generated by the modular lattice $\mathbf{M}_\kappa$ of height two with κ-many atoms [I]; and any proper subvariety of commutative binary modes [HB]. For further examples, see [B] and [HB].

7. Minimal Mal'cev varieties without idempotents

Corollary 6.2 described the lattice $\mathcal{L}_Q(\widetilde{\mathsf{V}})$ for V a locally finite, minimal Mal'cev variety in which every algebra has an idempotent. In this section we consider the alternate case in which every (equivalently, some) non-trivial member fails to have an idempotent. Our intent is to prove the following Theorem.

Theorem 7.1. *Let V be a locally finite, minimal, Mal'cev variety of Ω-algebras, and assume that some member of V has no idempotent element. Then the lattice of subquasivarieties of $\widetilde{\mathsf{V}}$ is*

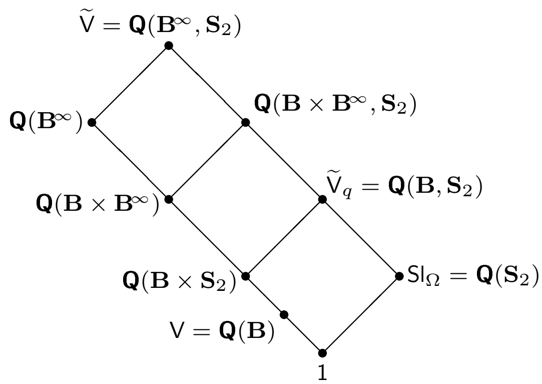

Figure 3

For the remainder of this section, let V be a variety satisfying the assumptions in the above Theorem. The structure of such a variety has been extensively studied. We summarize some of its properties below.

Theorem 7.2.

1. *V has (up to isomorphism) a unique subdirectly irreducible algebra $\mathbf{B}$. This algebra is quasiprimal, hence simple, and has no proper subalgebras.*
2. *$\mathbf{B}$ is V-prime.*
3. *V is a minimal quasivariety.*
4. *The congruence lattice of every finite algebra in V is Boolean.*

Proof. The first three of these were discussed in Section 6. Additional details can be found in [FM, Theorem 12.1]. The last follows from the fact that this variety is semisimple and arithmetical. $\square$

The proof of Theorem 7.1 operates by keeping track of idempotent elements in $\widetilde{\mathsf{V}}$-algebras. Let $B = \{b_1, \ldots, b_n\}$. Since $\mathbf{B}$ has no idempotent elements, for each $j \leq n$ there is a unary term τ_j such that $b_j\tau_j \neq b_j$. Let $\varepsilon(x)$ denote the following formula in the free variable x.

$$(\varepsilon(x)) \qquad x\tau_1 = x \;\&\; x\tau_2 = x \;\&\; \cdots \;\&\; x\tau_n = x.$$

Lemma 7.3. *Let $\mathbf{A} \in \widetilde{\mathsf{V}}$ and $a \in A$.*

(1) *The following are equivalent*
   - *(i)* a *is an idempotent element of* $\mathbf{A}$*;*
   - *(ii)* $\{a\}$ *is a fiber of* $\mathbf{A}$*;*
   - *(iii)* $\varepsilon(a)$ *holds in* $\mathbf{A}$.

(2) *If S is the semilattice replica of $\mathbf{A}$, then $\{\, s \in S : |A_s| = 1 \,\}$ is an ideal (downset) of S.*

Proof. From the construction of ε, we see that $\mathbf{B}$ satisfies the quasi-identity

$$\varepsilon(x) \;\rightarrow\; x = y.$$

Therefore, $\mathbf{Q}(\mathbf{B}) = \mathsf{V}$ satisfies that quasi-identity as well, which is to say, no non-trivial member of V contains an element that satisfies $\varepsilon(x)$. Now let A_s be the fiber of $\mathbf{A}$ containing a. Then $\mathbf{A}_s \in \mathsf{V}$, so, if $\varepsilon(a)$ holds, we must have $|A_s| = 1$, so *(iii)* implies *(ii)*. Since every fiber of $\mathbf{A}$ is a subalgebra, *(ii)* certainly implies *(i)*, and that *(i)* implies *(iii)* is trivial.

For the second claim, if $|A_s| = \{a\}$ and $s \geq t$ in S, then $a\varphi_{s,t}$ must be an idempotent element of $\mathbf{A}_t$, so $|A_t| = 1$. □

We now proceed much as we did in Sections 4 and 6. By Theorem 2.3, the regularization $\widetilde{\mathsf{V}}$ contains three subdirectly irreducible algebras: the algebra $\mathbf{B}$ described in Theorem 7.2, the two-element Ω-semilattice $\mathbf{S}_2$, and the algebra $\mathbf{B}^\infty$. $\mathbf{B}$ is isomorphic to a subalgebra of $\mathbf{B}^\infty$ (since it is one of the two fibers of $\mathbf{B}^\infty$). The crucial difference from the material in Section 6 is that $\mathbf{S}_2$ does not embed into $\mathbf{B}^\infty$, since the latter algebra has only one idempotent element.

Two other algebras will play a key role in our analysis: $\mathbf{B} \times \mathbf{S}_2$ and $\mathbf{B} \times \mathbf{B}^\infty$. The first is the Płonka sum of two copies of $\mathbf{B}$ over $\mathbf{S}_2$, with an isomorphism between the fibers. The second is the Płonka sum of the functor $F\colon S_2 \to \mathsf{V}$, in which $F(1) = \mathbf{B}^2$, $F(0) = \mathbf{B}$ and $F(0{\to}1)$ is one of the coordinate projection maps from $\mathbf{B}^2$ to $\mathbf{B}$. Letting $\{\infty\}$ denote the trivial fiber of $\mathbf{B}^\infty$, we compute $\mathbf{B} \times \mathbf{B}^\infty = \mathbf{B} \times (\mathbf{B} \uplus \{\infty\}) = \mathbf{B}^2 \uplus (\mathbf{B} \times \{\infty\})$. Thus we have an embedding of $\mathbf{B} \times \mathbf{S}_2$ into $\mathbf{B} \times \mathbf{B}^\infty$ given by: $(x, 1) \mapsto (x, x) \in B^2$ and $(x, 0) \mapsto (x, \infty) \in B \times \{\infty\}$. To summarize:

$$\mathbf{S}_2 \not\leq \mathbf{B}, \quad \mathbf{S}_2 \not\leq \mathbf{B}^\infty, \quad \mathbf{B} \leq \mathbf{B}^\infty, \quad \mathbf{B} \times \mathbf{S}_2 \leq \mathbf{B} \times \mathbf{B}^\infty. \tag{7–1}$$

Using (7–1), it is easy to verify that the inclusions illustrated in the Hasse diagram of Figure 3 are correct (it is enough to check the generators of each quasivariety). That $\mathsf{V} = \mathbf{Q}(\mathbf{B})$ and $\widetilde{\mathsf{V}} = \mathbf{SP}(\mathbf{S}_2, \mathbf{B}, \mathbf{B}^\infty) = \mathbf{Q}(\mathbf{S}_2, \mathbf{B}^\infty)$ follow from Theorems 2.3 and 7.2. To verify that the 9 quasivarieties shown in Figure 3 are pairwise distinct, it suffices to utilize the following (again, each of these can be verified simply by checking the generating algebras):

$$\begin{aligned}
\mathbf{Q}(\mathbf{B}^\infty) &\models \varepsilon(x) \;\&\; \varepsilon(y) \;\rightarrow\; x = y \\
\mathbf{Q}(\mathbf{B} \times \mathbf{B}^\infty, \mathbf{S}_2) &\models \varepsilon(x) \;\rightarrow\; \varepsilon(y) \\
\mathbf{Q}(\mathbf{B}, \mathbf{S}_2) &\models q_* \\
\mathbf{Q}(\mathbf{S}_2) &\models x * y = y * x \\
\mathbf{Q}(\mathbf{B}) &\models x * y = x.
\end{aligned} \tag{7–2}$$

Here, $x * y$ can be any convenient binary term witnessing the strong irregularity of V. One could take $x * y = xyy\tau$, where τ is a ternary discriminator term for V. As an example, we verify that $\mathbf{Q}(\mathbf{B} \times \mathbf{B}^\infty) \neq \mathbf{Q}(\mathbf{B}^\infty)$. First

$$\mathbf{B} \times \mathbf{B}^\infty \leq \mathbf{B}^\infty \times \mathbf{B}^\infty \in \mathbf{Q}(\mathbf{B}^\infty),$$

so it follows that

$$\mathbf{Q}(\mathbf{B} \times \mathbf{B}^\infty) \subseteq \mathbf{Q}(\mathbf{B}^\infty) \cap \mathbf{Q}(\mathbf{B} \times \mathbf{B}^\infty, \mathbf{S}_2).$$

On the other hand, $\mathbf{B}^\infty \not\models \varepsilon(x) \;\rightarrow\; \varepsilon(y)$ (since it has both an idempotent and a non-idempotent element), so we conclude that $\mathbf{Q}(\mathbf{B} \times \mathbf{B}^\infty) \subsetneq \mathbf{Q}(\mathbf{B}^\infty)$.

Notice that each of the formulas in (7–2) is equivalent to a finite set of quasi-identities. Thus Theorem 7.1, together with these quasi-identities, yields a finite basis (relative to $\widetilde{\mathsf{V}}$) for each subquasivariety.

To prove Theorem 7.1, all that remains is to show that every subquasivariety of $\widetilde{\mathsf{V}}$ is one of the nine in Figure 3. The proof proceeds via a sequence of seven easy lemmas.

For the remainder of this section, $\mathbf{A}$ will denote an arbitrary algebra in $\widetilde{\mathsf{V}}$, with semilattice replica S and fibers A_s, for $s \in S$. For $s \geq_. t$, $\varphi_{s,t} \colon A_s \to A_t$ denotes the Płonka homomorphism. We repeatedly use the following simple observation. For any two algebras $\mathbf{D}$ and $\mathbf{C}$, and nonempty set I, $\mathbf{D} \times \mathbf{C}$ can be embedded into $\mathbf{D} \times \mathbf{C}^I$ via the mapping $(d, c) \mapsto (d, c, c, \dots)$.

Lemma 7.4. *Let $\mathbf{A} \in \widetilde{\mathsf{V}}$ and let $\delta \in \operatorname{Con} \mathbf{A}$ be the congruence defined in* (3–1). *The following are equivalent:*

(1) $\delta \neq 1_A$;
(2) $\mathbf{B}$ *is a homomorphic image of* $\mathbf{A}$;
(3) $\mathbf{A}$ *has no idempotent elements.*

Proof. (1)⇒(2): If $\delta \neq 1_A$, then $\mathbf{A}/\delta$ is a nontrivial member of V. Since the only subdirectly irreducible algebra in V is $\mathbf{B}$, it follows that there are surjective homomorphisms $\mathbf{A} \twoheadrightarrow \mathbf{A}/\delta \twoheadrightarrow \mathbf{B}$.

(2)⇒(3): The image of an idempotent element is idempotent. Since $\mathbf{B}$ has no idempotents, neither does $\mathbf{A}$.

(3)⇒(1): Suppose $\delta = 1_A$. Let $s \in S$. If $|A_s| = 1$, we are done. If not, then, by the primeness of $\mathbf{B}$, $\mathbf{A}_s$ has a subalgebra $\mathbf{C}$ isomorphic to $\mathbf{B}$. Let a and b be distinct elements of C. Since $a \equiv b \pmod{\delta}$, there is $u < s$ such that $a\varphi_{s,u} = b\varphi_{s,u}$. But $\mathbf{C}$ is simple, so it follows that the image of C under $\varphi_{s,u}$ is a one-element subalgebra of $\mathbf{A}_u$. This element will be an idempotent of $\mathbf{A}$. ☐

Lemma 7.5. *Let $\mathbf{A} \in \widetilde{\mathsf{V}}$. If $\mathbf{A}$ has no idempotents, then $\mathbf{A} \in \mathbf{SP}(\mathbf{B} \times \mathbf{B}^\infty)$.*

Proof. By Lemma 7.4, $\mathbf{B}$ is a homomorphic image of $\mathbf{A}$. Since the only subdirectly irreducible algebras in $\widetilde{\mathsf{V}}$ are $\mathbf{S}_2$, $\mathbf{B}$ and $\mathbf{B}^\infty$, we have an embedding of $\mathbf{A}$ into ${\mathbf{B}^\infty}^{I'} \times \mathbf{B}^J \times \mathbf{S}_2^K$, for some sets I', J, K with $J \neq \varnothing$. Using the embeddings in (7–1) and setting $I = I' \cup J$ we have

$$\begin{aligned}\mathbf{A} \leq {\mathbf{B}^\infty}^{I'} \times \mathbf{B}^J \times \mathbf{S}_2^K \leq {\mathbf{B}^\infty}^I \times \mathbf{B}^I \times \mathbf{S}_2^K \leq {\mathbf{B}^\infty}^I \times \mathbf{B}^I \times \mathbf{B}^K \times \mathbf{S}_2^K \\ \leq (\mathbf{B} \times \mathbf{B}^\infty)^I \times (\mathbf{B} \times \mathbf{S}_2)^K \in \mathbf{SP}(\mathbf{B} \times \mathbf{B}^\infty). \quad ☐\end{aligned}$$

Lemma 7.6. *Let* $\mathbf{A}$ *be an algebra containing exactly one idempotent element. Then* $\mathbf{A} \in \mathsf{SP}(\mathbf{B}^\infty)$.

Proof. Since the trivial fibers induce an ideal on S, the minimum fiber $\mathbf{A}_0$, of $\mathbf{A}$, is the unique trivial fiber. For $t > 0$ in $\mathbf{S}$, let

$$\theta_t = \left\{ (x,y) \in A^2 : x = y \text{ or } (x \in A_s,\ y \in A_u,\ s \ngeq_{\cdot} t \text{ and } u \ngeq_{\cdot} t) \right\}.$$

It is easy to verify that θ_t is a congruence on $\mathbf{A}$ and that $\bigcap(\theta_t : t > 0) = 0_A$. Therefore, $\mathbf{A} \le \prod_{t>0} \mathbf{A}/\theta_t$. So it suffices to prove that each $\mathbf{A}/\theta_t \in \mathsf{SP}(\mathbf{B}^\infty)$.

Fix $t > 0$. The congruence θ_t acts like the identity on all fibers A_s for $s \ge t$, and collapses everything else to A_0. Thus the semilattice replica of $\mathbf{A}' = \mathbf{A}/\theta_t$ consists of a minimum element, 0, a unique atom, t, and other points above t. The fiber $\mathbf{A}'_0$ will be the unique trivial fiber of $\mathbf{A}'$.

Let $\mathbf{C}$ be the subalgebra of $\mathbf{A}'$ obtained by omitting A'_0. Then $\mathbf{A}' \cong \mathbf{C}^\infty$. Also, $\mathbf{C}$ has no idempotents, so by Lemma 7.5 can be embedded into $(\mathbf{B} \times \mathbf{B}^\infty)^I$ for some index set I.

Now $\mathbf{B} \times \mathbf{B}^\infty$ can be embedded into $(\mathbf{B}^\infty)^2$ in such a way that the image does not include the trivial fiber of $(\mathbf{B}^\infty)^2$. Take $J = I \times \{0,1\}$. It follows that $(\mathbf{B} \times \mathbf{B}^\infty)^I$ can be embedded into ${\mathbf{B}^\infty}^J$ excluding the trivial fiber, and therefore $[(\mathbf{B} \times \mathbf{B}^\infty)^I]^\infty$ can also be embedded into ${\mathbf{B}^\infty}^J$, now utilizing the trivial fiber of ${\mathbf{B}^\infty}^J$.

Finally, $\mathbf{C} \le (\mathbf{B} \times \mathbf{B}^\infty)^I$, so $\mathbf{A}/\theta_t \cong \mathbf{A}' \cong \mathbf{C}^\infty \le [(\mathbf{B} \times \mathbf{B}^\infty)^I]^\infty \le {\mathbf{B}^\infty}^J$ as desired. □

Lemma 7.7. *In* $\mathcal{L}_\mathcal{Q}(\widetilde{\mathsf{V}})$, $\mathbf{Q}(\mathbf{B}^\infty)$ *is covered by* $\widetilde{\mathsf{V}}$. *In fact,*

$$\mathsf{Q}(\mathbf{B}^\infty) = \left\{ \mathbf{A} \in \widetilde{\mathsf{V}} : \mathbf{S}_2 \nleq \mathbf{A} \right\}.$$

Proof. Observe that the second claim implies the first: if Z is a quasivariety and $\mathsf{Q}(\mathbf{B}^\infty) \subsetneq \mathsf{Z} \subseteq \widetilde{\mathsf{V}}$, then there is an algebra $\mathbf{A} \in \mathsf{Z} - \mathsf{Q}(\mathbf{B}^\infty)$. Consequently, $\mathbf{S}_2 \in \mathsf{S}(\mathbf{A}) \subseteq \mathsf{Z}$ and $\mathbf{B}^\infty \in \mathsf{Z}$, so $\mathsf{Z} \supseteq \mathsf{SP}(\mathbf{S}_2, \mathbf{B}^\infty) \supseteq \mathsf{SP}(\mathbf{S}_2, \mathbf{B}, \mathbf{B}^\infty) = \widetilde{\mathsf{V}}$.

For the second claim, from the formulas in (7–2),

$$\mathsf{Q}(\mathbf{B}^\infty) \vDash \varepsilon(x) \ \ \& \ \ \varepsilon(y) \ \rightarrow \ x = y$$

Therefore no member of $\mathsf{Q}(\mathbf{B}^\infty)$ can contain a subalgebra isomorphic to $\mathbf{S}_2$.

Conversely, suppose $\mathbf{A}$ is an algebra that fails to extend $\mathbf{S}_2$. In other words, $\mathbf{A}$ has at most one idempotent element. If $\mathbf{A}$ has an idempotent, then by Lemma 7.6, $\mathbf{A} \in \mathsf{Q}(\mathbf{B}^\infty)$. If $\mathbf{A}$ has no idempotents, then by Lemma 7.5, $\mathbf{A} \in \mathsf{SP}(\mathbf{B} \times \mathbf{B}^\infty) \subseteq \mathsf{Q}(\mathbf{B}^\infty)$, since $\mathbf{B} \le \mathbf{B}^\infty$. □

Lemma 7.8. *In* $\mathcal{L}_\mathcal{Q}(\widetilde{\mathsf{V}})$, $\mathbf{Q}(\mathbf{B} \times \mathbf{B}^\infty, \mathbf{S}_2)$ *is covered by* $\widetilde{\mathsf{V}}$. *In fact,*

$$\mathsf{Q}(\mathbf{B} \times \mathbf{B}^\infty, \mathbf{S}_2) = \left\{ \mathbf{A} \in \widetilde{\mathsf{V}} : \mathbf{B}^\infty \nleq \mathbf{A} \right\} = \mathsf{Q}(\mathbf{B} \times \mathbf{B}^\infty) \cup \mathsf{SI}_\Omega.$$

Proof. Using an argument similar to that of the previous Lemma together with the relationships in (7–2), one obtains the first equality. Thus it suffices to prove that $\left\{ \mathbf{A} \in \widetilde{\mathsf{V}} : \mathbf{B}^\infty \nleq \mathbf{A} \right\} \subseteq \mathsf{Q}(\mathbf{B} \times \mathbf{B}^\infty) \cup \mathsf{SI}_\Omega$.

So suppose that $\mathbf{A} \in \widetilde{\mathsf{V}} - \mathsf{Sl}_\Omega$ and $\mathbf{B}^\infty \not\leq \mathbf{A}$. Then $\mathbf{A}$ must have a nontrivial fiber $\mathbf{A}_s$. By the primeness of $\mathbf{B}$, $\mathbf{A}_s$ has a subalgebra $\mathbf{C}$ isomorphic to $\mathbf{B}$. If $\mathbf{A}$ has an idempotent element, then $\mathbf{A}$ has a trivial fiber, $A_t = \{0_t\}$, with $t < s$ (since the trivial fibers form an ideal of S). But then $C \cup A_t$ forms a subalgebra of $\mathbf{A}$ isomorphic to $\mathbf{B}^\infty$, which is a contradiction. Therefore $\mathbf{A}$ has no idempotents. By Lemma 7.5, $\mathbf{A} \in \mathbf{Q}(\mathbf{B} \times \mathbf{B}^\infty)$. □

Lemma 7.9. $\mathbf{Q}(\mathbf{B}^\infty) \cap \mathbf{Q}(\mathbf{B} \times \mathbf{B}^\infty, \mathbf{S}_2) = \mathbf{Q}(\mathbf{B} \times \mathbf{B}^\infty)$.

Proof. Since every non-trivial member of Sl_Ω extends $\mathbf{S}_2$, we have from Lemma 7.7 that $\mathbf{Q}(\mathbf{B}^\infty) \cap \mathsf{Sl}_\Omega = \mathbf{1}$. The result now follows from Lemma 7.8.

Lemma 7.10. *$\widetilde{\mathsf{V}}_q$ is the largest subquasivariety of $\widetilde{\mathsf{V}}$ that omits $\mathbf{B} \times \mathbf{B}^\infty$.*

Proof. Since $\mathbf{B} \times \mathbf{B}^\infty$ has a Płonka homomorphism that is not injective, $\mathbf{B} \times \mathbf{B}^\infty \notin \widetilde{\mathsf{V}}_q$ by Proposition 3.3. Let $\mathsf{Z} \in \mathcal{L}_Q(\widetilde{\mathsf{V}})$ and assume Z omits $\mathbf{B} \times \mathbf{B}^\infty$. Suppose there is an algebra $\mathbf{A} \in \mathsf{Z} - \widetilde{\mathsf{V}}_q$. Then $\mathbf{A}$ has two fibers A_s, A_t with $s >_{\cdot} t$ and $\varphi_{s,t}$ not injective. Let a and b be distinct elements of A_s such that $a\varphi_{s,t} = b\varphi_{s,t}$, and let C be the subuniverse of $\mathbf{A}_s$ generated by $\{a, b\}$. Since V is locally finite, $1 < |C| < \infty$. By Theorem 7.2(4), $\mathrm{Con}(\mathbf{C})$ is a Boolean lattice.

Let $\psi = \varphi_{s,t}{\restriction_C}$ and let $\theta = \ker\psi$. Suppose $\theta = 1_C$. Since $\mathbf{B}$ is V-prime, $\mathbf{C}$ has a subalgebra $\mathbf{B}'$ isomorphic to $\mathbf{B}$. Therefore, $B' \cup \psi(B')$ is a subuniverse of $\mathbf{A}$ isomorphic to $\mathbf{B}^\infty$. But then $\mathbf{B} \times \mathbf{B}^\infty \leq (\mathbf{B}^\infty)^2 \in \mathsf{Z}$, contradicting our assumptions about Z.

Therefore, $\theta \neq 1_C$. And $(a, b) \in \theta$ implies $\theta \neq 0_C$. Let ν denote the complement of θ in $\mathrm{Con}\,\mathbf{C}$. We have

$$0 < \nu < 1, \quad \theta \cap \nu = 0, \quad \theta \circ \nu = 1.$$

Thus $\mathbf{C} \cong \mathbf{C}/\theta \times \mathbf{C}/\nu$, and both of $\mathbf{C}/\theta$ and $\mathbf{C}/\nu$ are non-trivial. By the definition of θ, $\mathbf{C}\psi \cong (\mathbf{C}/\nu)\psi$ is also non-trivial. Again, appealing to primeness, each of $\mathbf{C}/\theta$, $\mathbf{C}/\nu$ and $\mathbf{C}\psi$ have subalgebras isomorphic to $\mathbf{B}$. It follows that $\mathbf{C} \uplus (\mathbf{C}\psi)$ has a subalgebra isomorphic to $\mathbf{B}^2 \uplus \mathbf{B}$. But this latter algebra is isomorphic to $\mathbf{B} \times \mathbf{B}^\infty$, which once again contradicts our assumptions on Z. □

Proof of Theorem 7.1. We observed earlier that the nine quasivarieties illustrated in Figure 3 are all distinct. Let $\mathsf{Z} \in \mathcal{L}_Q(\widetilde{\mathsf{V}})$. We wish to prove that Z is one of the points in the figure.

Let $\mathsf{K} = \mathsf{Z} \cap \{\mathbf{B}, \mathbf{S}_2, \mathbf{B}^\infty\}$. If $\mathbf{B} \notin \mathsf{K}$, then every member of Z consists of nothing but trivial fibers—that is to say, Z is a collection of semilattices. As we have already remarked, Sl_Ω is a minimal quasivariety, so either $\mathsf{Z} = \mathsf{Sl}_\Omega$ or $\mathsf{Z} = \mathbf{1}$.

We are now reduced to the following four cases:

(1) $\mathsf{K} = \{\mathbf{B}, \mathbf{S}_2, \mathbf{B}^\infty\}$;
(2) $\mathsf{K} = \{\mathbf{B}, \mathbf{B}^\infty\}$;
(3) $\mathsf{K} = \{\mathbf{B}, \mathbf{S}_2\}$;
(4) $\mathsf{K} = \{\mathbf{B}\}$.

In case (1), $\mathsf{Z} = \mathbf{SP}(\mathbf{B}, \mathbf{S}_2, \mathbf{B}^\infty) = \widetilde{\mathsf{V}}$. In case (2), $\mathbf{B}^\infty \in \mathsf{Z}$ but $\mathbf{S}_2 \notin \mathsf{Z}$, so by Lemma 7.7, $\mathsf{Z} = \mathbf{Q}(\mathbf{B}^\infty)$.

Now consider case (3). By Proposition 3.3 and Lemma 7.8,

$$\widetilde{\mathsf{V}}_q \subseteq \mathsf{Z} \subseteq \mathbf{Q}(\mathbf{B} \times \mathbf{B}^\infty, \mathbf{S}_2).$$

If $\mathbf{B} \times \mathbf{B}^\infty \in \mathsf{Z}$, then we have $\mathsf{Z} = \mathbf{Q}(\mathbf{B} \times \mathbf{B}^\infty, \mathbf{S}_2)$. If not, then by Lemma 7.10, $\mathsf{Z} = \widetilde{\mathsf{V}}_q$.

Finally, assume that $\mathsf{K} = \{\mathbf{B}\}$. By Lemmas 7.7, 7.8 and 7.9, $\mathbf{Q}(\mathbf{B}) \subseteq \mathsf{Z} \subseteq \mathbf{Q}(\mathbf{B} \times \mathbf{B}^\infty)$. If $\mathbf{B} \times \mathbf{B}^\infty \in \mathsf{Z}$, then $\mathsf{Z} = \mathbf{Q}(\mathbf{B} \times \mathbf{B}^\infty)$. So assume that $\mathbf{B} \times \mathbf{B}^\infty$ is not a member of Z. By Lemma 7.10, $\mathsf{Z} \subseteq \widetilde{\mathsf{V}}_q$, and therefore every Płonka homomorphism is injective.

Recall that $\mathsf{V} = \mathbf{Q}(\mathbf{B})$. Suppose that $\mathsf{V} \neq \mathsf{Z}$. To conclude the proof, we must show that $\mathsf{Z} = \mathbf{Q}(\mathbf{B} \times \mathbf{S}_2)$. Let $\mathbf{A} \in \mathsf{Z} - \mathsf{V}$. Then $\mathbf{A}$ must have at least two fibers A_s and A_t with $s > t$. These fibers must be non-trivial since $\mathbf{S}_2 \notin \mathsf{Z}$. By primeness, each of $\mathbf{A}_s$ and $\mathbf{A}_t$ must contain a subalgebra isomorphic to $\mathbf{B}$. Therefore, by the injectivity of $\varphi_{s,t}$, $\mathbf{B} \uplus \mathbf{B} \cong \mathbf{B} \times \mathbf{S}_2$ is a subalgebra of $\mathbf{A}$, so $\mathbf{Q}(\mathbf{B} \times \mathbf{S}_2) \subseteq \mathsf{Z}$.

For the converse, let $\mathbf{A}$ be an arbitrary member of Z. Since $\mathbf{A} \in \widetilde{\mathsf{V}}_q$, by Theorem 3.5 we have $\mathbf{A} \leq \mathbf{B}^I \times \mathbf{S}_2^J$, for some sets I, J. We must have I nonempty since otherwise $\mathbf{A} \in \mathsf{Sl}_\Omega$, which would imply that $\mathbf{S}_2 \in \mathsf{Z}$ which is false. Therefore

$$\mathbf{A} \leq \mathbf{B}^I \times \mathbf{S}_2^J \leq (\mathbf{B} \times \mathbf{S}_2)^K \in \mathbf{Q}(\mathbf{B} \times \mathbf{S}_2)$$

where $K = I \cup J$. □

Example 7.11. The variety BA of Boolean algebras satisfies the hypotheses of Theorem 7.1. The prime algebra $\mathbf{B}$ is, of course, the two-element Boolean algebra. To regularize BA we need an axiomatization not involving constant operations, as was done in [P4]. Boolean algebras were axiomatized there as algebras $\langle B, \wedge, \vee, ' \rangle$ satisfying the axioms of a distributive lattice, together with the following

$$(x \vee y)' = x' \wedge y', \qquad x'' = x, \qquad x \vee (x \wedge x') = x \tag{7–3}$$

$$x \wedge x' = y \wedge y'. \tag{7–4}$$

The regularization $\widetilde{\mathsf{BA}}$ of BA may then be defined by the axioms of distributive bisemilattices, the identities (7–3) and the identity

$$(x \wedge x') \vee (y \wedge y') = x \wedge x' \wedge y \wedge y'.$$

The three-element algebra $\mathbf{B}^\infty$ has appeared in the literature under various names such as "the weak extension of Boolean logic" and the "Bochvar system of logic" [Gu], [Be]. The formula $\varepsilon(x)$ can be taken to be $x = x'$. The lattice of subquasivarieties of $\widetilde{\mathsf{BA}}$ is given in Figure 3, and an axiomatization of each quasivariety can be obtained from (7–2). Also, the quasi-identity q_* can be replaced by the following:

$$x \leq_\wedge y \leq_\wedge z \ \& \ y \leq_\vee z \leq_\vee x \ \rightarrow \ y = z.$$

Example 7.12. In example 6.5 we considered the minimal varieties of rings "without identity". If we consider rings as algebras $\langle R, +, -, \cdot, 1 \rangle$ where 1 denotes a unary operation satisfying the axioms $(x)1 \cdot y = y = y \cdot (x)1$, we get somewhat different behavior. This variety does not contain any non-trivial zero-rings, so the minimal varieties are generated by the fields $\mathbb{Z}_p$. But 0 is no longer an idempotent element. Therefore, we get a description of the subquasivarieties of the regularization by appealing to Theorem 7.1 instead of 6.2. For $\varepsilon(x)$ we can use $x = x + (x)1$. Of course, the Boolean algebra example just above is a special case of this one with $p = 2$.

References

[B] C. Bergman, *Structural completeness in algebra and logic*, Colloq. Math. Soc. J. Bolyai **54** (1988), 59–73.

[Be] J. Berman, *Free spectra of 3-element algebras*, Springer Lecture Notes in Mathematics **1004** (1983), 10–53.

[BM] C. Bergman, R. McKenzie, *Minimal varieties and quasivarieties*, J. Austral. Math. Soc. (Series A) **48** (1990), 133–147.

[BS] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer-Verlag, New York, 1981.

[CK] D. M. Clark and P. H. Krauss, *Para primal algebras*, Algebra Universalis **6** (1976), 165–192.

[CM] B. Csákány, L. Megyesi, *Varieties of idempotent medial quasigroup*, Acta Sci. Math. **37** (1975), 17–23.

[Cs] B. Csákány, *Varieties of affine modules*, Acta Sci. Math. **37** (1975), 3–10.

[DG] J. Dudek, E. Graczyinska, *The lattice of varieties of algebras*, Bull. Acad. Polon. Sci. Ser. Sci. Math **29** (1981), 337–340.

[DMS] B. A. Davey, K. R. Miles and V. J. Schumann, *Quasi-identities, Mal'cev conditions and congruence regularity*, Acta Sci. Math. **51** (1987), 39–55.

[FM] R. Freese and R. McKenzie, *Commutator Theory for Congruence Modular Varieties*, London Mathematical Society Lecture Notes no. 125, Cambridge University Press, Cambridge, 1987.

[G] G. Grátzer, *Universal Algebra,* Second edition, Springer-Verlag, New York, 1979.

[GR] G. Gierz, A. Romanowska, *Duality for distributive bisemilattices*, J. Austral. Math. Soc. (Series A) **51** (1991), 247–275.

[Gu] F. Guzmán, *Three-valued logic in the semantics of programming languages* (Preprint 1992).

[HB] L. Hogben, C. Bergman, *Deductive varieties of modules and universal algebras*, Trans. Amer. Math. Soc. **289** (1985), 303–320.

[I] V.I. Igošin, *Quasivarieties of lattices*, Mat. Z. **16** (1974), 49–56 (Russian); English Transl., Math. Notes **16** (1974), 613–617.

[JK] J. Jeźek, T. Kepka, *The lattice of varieties of commutative idempotent abelian distributive groupoids*, Algebra Universalis **5** (1975), 225–237.

[K] K. Kearnes, *Idempotent simple algebras* (Preprint, 1994).

[Ka] J. A. Kalman, *The join of the varieties of implication algebras and semilattices*, Math. Chronicle **9** (1980), 41–52.

[KS] J. Kalicki, D. Scott, *Equational completeness of abstract algebras*, Indag. Math. **17** (1955), 650–659.

[L] L. Libkin, *Towards a theory of edible powerdomains* (Preprint, 1993).

[LPP] H. Lakser, R. Padmanabhan, C.R. Platt, *Subdirect decomposition of Płonka sums*, Duke Math. Journal **39** (1972), 485–488.

[Ma] A.I. Mal'cev, *Algebraic Systems*, Springer Verlag, Berlin, 1973.

[Me] I.I. Mel'nik, *Normal closures of perfect varieties of universal algebras, Ordered sets and Lattices*, lzdat. Saratov. Univ., Saratov, (1971), 56–65. (Russian)

[Mi] A. Mitschke, *Summen von gerichteten Systemen von Algebren*, Mitt. Math. Sem. Giesssen **124** (1977), 1–56.

[MMT] R. McKenzie, G. McNulty, T. Taylor, *Algebras, Lattices, Varieties,*, vol. 1, Wadsworth, Monterey, 1987.

[OS] F. Osterman, J. Schmidt, *Der baryzentrische Kalkul als axiomatische Grundlage der affinen Geometrie*, J. Reine Angew. Math. **224** (1966), 44–57.

[P1] J. Płonka, *On a method of construction of abstract algebras*, Fund. Math. **61** (1967), 183–189.

[P2] ______, *On distributive quasilattices*, Fund. Math. **60** (1967), 191–200.

[P3] ______, *On equational classes of abstract algebraas defined by regular equations*, Fund. Math. **64** (1969), 241–247.

[P4] ______, *On sums of direct systems of Boolean algebras*, Coll. Math. **20** (1969), 209–214.

[PR] J. Płonka, A. Romanowska, *Semilattice sums*, Universal Algebra and Quasigroup Theory (A. Romanowska, J.D.H. Smith, eds.), Heldermann Verlag, Berlin, 1992, pp. 123–158.

[PRR] A. Pilitowska, A. Romanowska, B. Roszkowska, *Products of mode varieties and algebras of subalgebras* (Preprint, 1994).

[PRS] A. Pilitowska, A. Romanowska, J.D.H. Smith, *Affine spaces and algebras of subalgebras* (Preprint, 1992).

[Pu] H. Puhlmann, *The snack power domain for database semantics*, MFCS-93, pp. 650–659.

[Q] R. W. Quackenbush, *Algebras with minimal spectrum*, Algebra Universalis **10** (1980), 117–129.

[R] A. Romanowska, *On regular and regularized varieties*, Algebra Universalis **23** (1986), 215–241.

[Ro] B. Roszkowska, *On some varieties of symmetric idempotent entropic groupoids*, Universal and Applied Algebra, (K. Hałkowska, B. Stawski, eds.), World Scientific, 1989, pp. 254–274.

[RS1] A. Romanowska, J.D.H. Smith, *From affine to projective geometry via convexity*, Springer Lecture Notes in Mathematics **1149** (1985), 255–270.

[RS2] ———, *Modal Theory*, Heldermann Verlag, Berlin, 1985.

[RS3] ———, *On the structure of semilattice sums*, Czechoslovak J. Math **41** (1991), 24–43.

[RS4] ———, *Duality for semilattice representations* (Preprint, 1993).

[Sz1] Á. Szendrei, *On idempotent reducts of modules I, II*, Colloq. Math. Soc. J. Bolyai **29** (1982), 753–768, 769–780.

[Sz2] ———, *Strictly simple algebras and minimal varieties*, Universal Algebra and Quasigroup Theory (A. Romanowska, J.D.H. Smith,, eds.), Heldermann Verlag, Berlin, 1992, pp. 209–239.

[Sz3] ———, *Simple surjective algebras having no proper subalgebras*, J. Australian Math. Soc. (Series A) **48** (1990), 434–454.

[T] A. Tarski, *Equationally complete rings and relation algebras*, Indag. Math. **18** (1956), 39–46.

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