THE AMALGAMATION CLASS OF A DISCRIMINATOR VARIETY IS FINITELY AXIOMATIZABLE

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Discriminator varieties have been extensively studied since their introduction by Pixley in 1970. Among their attributes, discriminator varieties exhibit a strong relationship between the quantifier-free formulas and certain terms in the language of the variety. This paper exploits that relationship in order to prove that some important classes of algebras, generally defined using algebraic properties, can be described by a finite set of first-order sentences.

If $\mathcal{K}$ is a class of algebras, define $\boldsymbol{Ps}(\mathcal{K})$ to be the class of all algebras isomorphic to a subdirect product of members of $\mathcal{K}$. An algebra $\mathbf{A}$ of $\mathcal{K}$ is an *amalgamation base* of $\mathcal{K}$ if, for every $\mathbf{B}_0, \mathbf{B}_1 \in \mathcal{K}$ and α_0, α_1 embedding $\mathbf{A}$ into $\mathbf{B}_0$ and $\mathbf{B}_1$ respectively, there is an algebra $\mathbf{C}$ of $\mathcal{K}$ and embeddings β_0 of $\mathbf{B}_0$ into $\mathbf{C}$ and β_1 of $\mathbf{B}_1$ into $\mathbf{C}$ such that $\beta_1 \circ \alpha_1 = \beta_0 \circ \alpha_0$. The collection of all amalgamation bases is called the *amalgamation class,* $\operatorname{Amal}(\mathcal{K})$.

Let $\mathcal{V}$ be a finitely generated discriminator variety of finite type. In this paper, the following classes will be shown to be finitely axiomatizable:

(1) $\boldsymbol{Ps}(\mathcal{S})$, where $\mathcal{S}$ is a set of simple algebras of $\mathcal{V}$;
(2) $\operatorname{Amal}(\mathcal{V})$.

These results are taken from the author's doctoral thesis written under Ralph McKenzie. The author is greatly indebted to him for his guidance and suggestion of this problem.

Notation. For a class $\mathcal{K}$ of algebras, $\mathcal{K}_{\mathrm{si}}$ denotes the class of subdirectly irreducible algebras of $\mathcal{K}$. Terms in the language of $\mathcal{K}$ will be denoted by lower case Greek letters. If α is a term and $\mathbf{A} \in \mathcal{K}$ then $\alpha^{\mathbf{A}}$ denotes the term function of $\mathbf{A}$ corresponding to α. For convenience, the sequence of letters $x_0, x_1, \dots, x_{n-1}$ will often be abbreviated $\vec{x}$.

The lattice of congruences of an algebra $\mathbf{A}$ is denoted by $\operatorname{Con}(\mathbf{A})$ and has smallest element Δ and largest element ∇. If $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and $\Theta \in \operatorname{Con}\mathbf{A}$, then $\Theta\restriction_B$ denotes the congruence $\Theta \cap B^2$ of $\mathbf{B}$. For definitions and basic facts not explained here, the reader can consult [2] or [3].

Definition 1. A variety $\mathcal{V}$ is a *discriminator variety* if there is a term $\sigma(x,y,z)$ in the language of $\mathcal{V}$ (called the *discriminator term*) such that an algebra $\mathbf{A} \in \mathcal{V}$ is subdirectly irreducible or trivial if and only if

$$\mathbf{A} \models \big(\sigma(x,x,z) \approx z \ \wedge\ x \not\approx y \to \sigma(x,y,z) \approx x\big).$$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

In [6] Pixley showed that a finitely generated variety is a discriminator variety if and only if it is arithmetical, and every subalgebra of a subdirectly irreducible algebra is either simple or trivial.

Suppose $(\mathbf{A}_i : i \in I)$ is a family of algebras and D is an ultrafilter on the set I. Then D induces a congruence (also denoted D) on $\prod(\mathbf{A}_i : i \in I)$ by $a \equiv b \pmod{\mathsf{D}}$ if and only if $\{\, i \in I : a_i = b_i \,\} \in \mathsf{D}$.

To build a set of sentences describing $\boldsymbol{Ps}(\mathcal{S})$, we first need sentences to insure that the discriminator term has the desired properties. These were discovered by R. McKenzie [5].

Theorem 2. *Let $\mathcal{L}$ be a first-order language with no non-logical relation symbols and with function symbols $\{\, f_i : i < k \,\}$, k a cardinal. Suppose $\sigma(x, y, z)$ is a term of $\mathcal{L}$ and let Σ be the set consisting of the following identities*

- *(e0)* $\sigma(x, x, y) \approx y$
- *(e1)* $\sigma(x, y, x) \approx x$
- *(e2)* $\sigma(x, y, y) \approx x$
- *(e3)* $\sigma(x, \sigma(x, y, z), y) \approx y$
- *(e4)$_i$* $\sigma(x, y, f_i(v_0, v_1, \ldots, v_{n_i-1})) \approx \sigma\big(x, y, f_i[\sigma(x, y, v_0), \ldots, \sigma(x, y, v_{n_i-1})]\big)$

for $i < k$, where f_i is n_i-ary.

1. *The variety $\mathcal{V}$ determined by the equations Σ is a discriminator variety with discriminator term σ.*
2. *Every finite algebra of $\mathcal{V}$ is a direct product of simple algebras.*
3. *For every $\mathbf{A} \in \mathcal{V}$ and $a, b \in A$, the binary relation*

$$\Theta(a, b) := \big\{\, (x, y) \in \mathbf{A}^2 : \sigma^{\mathbf{A}}(a, b, x) = \sigma^{\mathbf{A}}(a, b, y) \,\big\}$$

   is the smallest congruence of $\mathbf{A}$ containing (a, b).

4. *For every quantifier free formula ϕ of $\mathcal{L}$, there are terms α and β of $\mathcal{L}$ such that for every $\mathbf{B} \in \mathcal{V}_{\mathrm{si}}$ and $b_0, \ldots, b_{m-1} \in B$,*

$$\mathbf{B} \vDash \phi(\vec{b}) \ \leftrightarrow \ (\exists y)\ \alpha(\vec{b}, y) \not\approx \beta(\vec{b}, y).$$

Corollary 3. *Let $\mathcal{V}$ be a discriminator variety with discriminator term σ. Then the equations of Σ hold in $\mathcal{V}$. Let ϕ, α and β be as in Theorem 2(4). For any $\mathbf{A} \in \mathcal{V}$ and $a_0, \ldots, a_{m-1} \in A$, the following are equivalent:*

1. *there exists a coatom Ψ of* $\operatorname{Con}\mathbf{A}$ *such that $\mathbf{A}/\Psi \vDash \phi(a_0/\Psi, \ldots a_{m-1}/\Psi)$;*
2. $\mathbf{A} \vDash (\exists y)\ \alpha(a_0, \ldots, a_{m-1}, y) \not\approx \beta(a_0, \ldots, a_{m-1}, y)$.

Proof. Let $\mathbf{A} \in \mathcal{V}$ and suppose (i) holds. Take $\mathbf{B} = \mathbf{A}/\Psi$ and $b_i = a_i/\Psi$ for $i < m$. Then $\mathbf{B} \vDash \phi(b_0, \ldots, b_{m-1})$ implies by Theorem 2 that there is $z \in B$ with $\alpha^{\mathbf{B}}(b_0, \ldots, b_{m-1}, z) \neq \beta^{\mathbf{B}}(b_0, \ldots, b_{m-1}, z)$. Choose $y \in A$ with $y/\Psi = z$. Since $\alpha^{\mathbf{A}}(\vec{a}, y)/\Psi = \alpha^{\mathbf{B}}(\vec{b}, z) \neq \beta^{\mathbf{B}}(\vec{b}, z) = \alpha^{\mathbf{B}}(\vec{a}, y)/\Psi$ we conclude (ii).

Conversely, if α and β disagree for some $y \in A$, then they are separated by a completely meet-irreducible congruence Ψ. By semi-simplicity, $\mathbf{B} = \mathbf{A}/\Psi$ is simple, and we reverse the implication above to derive (i). $\Box$

For the remainder of this paper, suppose $\mathcal{V}$ is a finitely generated discriminator variety of finite type, that is the language of $\mathcal{V}$ has only finitely many basic operation and constant symbols. If $\mathbf{B}$ is a finite structure of this language, then there is a quantifier-free formula $\mathrm{Dg}_{\mathbf{B}}(x_0, x_1, \ldots, x_{n-1})$ (the "diagram of $\mathbf{B}$") such that, for every structure $\mathbf{A}$ and $a_0, a_1, \ldots, a_{n-1} \in A$, $\mathbf{A} \models \mathrm{Dg}_{\mathbf{B}}(\vec{a})$ if and only if $\{a_0, a_1, \ldots, a_{n-1}\}$ is the universe of a subalgebra of $\mathbf{A}$ isomorphic to $\mathbf{B}$. Setting ϕ equal to $\mathrm{Dg}_{\mathbf{B}}$ in the Corollary, we obtain terms α and β and elements $a_0, a_1, \ldots, a_{n-1}, b$ of A with $\alpha^{\mathbf{A}}(\vec{a}, b) \neq \beta^{\mathbf{A}}(\vec{a}, b)$ if and only if, for some coatom Ψ of $\mathrm{Con}\,\mathbf{A}$, $\mathbf{A}/\Psi$ contains a copy of $\mathbf{B}$ as a subalgebra. What is more surprising, we can strengthen this inequation in such a way that $\mathbf{A}/\Psi$ will be isomorphic to $\mathbf{B}$. This idea is incorporated into the following theorem.

Theorem 4. *Let $\mathcal{V}$ be a finitely generated discriminator variety of finite type. Let $\mathcal{S} \subseteq \mathcal{V}_{\mathrm{si}}$. Then $\boldsymbol{Ps}(\mathcal{S})$ is a finitely axiomatizable class.*

Proof. Since $\mathcal{V}$ is finitely generated, we may assume that $\mathcal{S}$ is a finite set of finite algebras, $\mathcal{S} = \{\mathbf{L}_0, \ldots, \mathbf{L}_{m-1}\}$. Since the language is of finite type, the set Σ of Theorem 2 is finite. Informally, we need to add to Σ a sentence saying that for any pair of distinct elements c and d, there is a completely meet-irreducible congruence Ψ separating c and d so that the quotient algebra modulo Ψ is isomorphic to one of the $\mathbf{L}_j$s.

Fix $j < m$. Let $\mathbf{L} = \mathbf{L}_j$ and $r = \mathrm{card}(L)$. By Corollary 3, there are terms α and β such that for every $\mathbf{A} \models \Sigma$ and $a_0, a_1, \ldots, a_{r-1}, c, d \in A$

$$\mathbf{A} \models (\exists y)\ \alpha(\vec{a}, y, c, d) \not\approx \beta(\vec{a}, y, c, d)$$

if and only if there is a coatom Ψ of $\mathrm{Con}\,\mathbf{A}$ such that

$$\mathbf{A}/\Psi \models c/\Psi \not\approx d/\Psi\ \wedge\ \mathrm{Dg}_{\mathbf{L}}(a_0/\Psi, \ldots, a_{r-1}/\Psi).$$

Now define the formula $\mathrm{Sep}_{\mathbf{L}}(u, v)$ to be

$$(\exists x_0, x_1, \ldots, x_{r-1})(\exists y)(\forall z)\ \Big[\alpha(\vec{x}, y, u, v) \not\approx \beta(\vec{x}, y, u, v)\ \wedge \\ \bigvee_{i<r} \big(\sigma(x_i, z, \alpha(\vec{x}, y, u, v)) \not\approx \sigma(x_i, z, \beta(\vec{x}, y, u, v))\big)\Big].$$

The key claim is that for any $\mathbf{A} \models \Sigma$ and $c, d \in A$, $\mathbf{A} \models \mathrm{Sep}_{\mathbf{L}}(c, d)$ if and only if there is a coatom Ψ of $\mathrm{Con}\,\mathbf{A}$ such that $c/\Psi \neq d/\Psi$ and $\mathbf{A}/\Psi \cong \mathbf{L}$. Once this is established, the members of Σ together with the sentence

$$(\forall u)(\forall v)\big(u \not\approx v\ \rightarrow\ \bigvee_{j<m} \mathrm{Sep}_{\mathbf{L}_j}(u, v)\big)$$

will axiomatize $\boldsymbol{Ps}(\mathcal{S})$.

Suppose first that for $c, d \in A$, there is $\Psi \in \operatorname{Con}\mathbf{A}$ such that $c/\Psi \neq d/\Psi$ and $\mathbf{A}/\Psi \cong \mathbf{L}$. Then $\mathbf{A}/\Psi$ satisfies $\mathrm{Dg}_{\mathbf{L}}$ for some elements $g_0, g_1, \ldots, g_{r-1}$. Choosing a_i in A to represent g_i modulo Ψ, there is a $b \in A$ such that $\alpha^{\mathbf{A}}(\vec{a}, b, c, d) \neq \beta^{\mathbf{A}}(\vec{a}, b, c, d)$. Now let e be any element of A. Since $A/\Psi = \{g_0, g_1, \ldots, g_{r-1}\}$, there is $i < r$ such that $e \equiv a_i \pmod{\Psi}$. For this i we have $\sigma^{\mathbf{A}}(a_i, e, \alpha^{\mathbf{A}}(\vec{a}, b, c, d)) \neq \sigma^{\mathbf{A}}(a_i, e, \beta^{\mathbf{A}}(\vec{a}, b, c, d))$ since they are incongruent modulo Ψ. Thus $\mathbf{A}$ satisfies $\mathrm{Sep}_{\mathbf{L}}(c, d)$.

Conversely, let $\mathbf{A} \models \mathrm{Sep}_{\mathbf{L}}(c, d)$. Let $a_0, a_1, \ldots, a_{r-1}, b$ be elements that witness the existential quantifiers. Denote the elements $\alpha^{\mathbf{A}}(\vec{a}, b, c, d)$ and $\beta^{\mathbf{A}}(\vec{a}, b, c, d)$ by α and β respectively. Write $\mathbf{A}$ as a subdirect product of subdirectly irreducible algebras, $\mathbf{A} \leq \prod(\mathbf{A}_t : t \in T)$, and let Θ_t be the kernel of the projection $\mathbf{A} \twoheadrightarrow \mathbf{A}_t$. Set $U = \{\, t \in T : \alpha \not\equiv \beta \pmod{\Theta_t} \,\}$ and for every $x \in A$, $V_x = \{\, t \in T : \text{for some } i < r,\ a_i \equiv x \pmod{\Theta_t} \,\}$.

Claim. *There is an ultrafilter* D *on* T *such that* $U \in \mathsf{D}$ *and for every* $x \in A$, $V_x \in \mathsf{D}$.

Once the claim is established the Theorem will follow easily. For, take $\Psi = \mathsf{D}\restriction_A$. Ψ is a coatom of $\operatorname{Con}\mathbf{A}$ since $\mathbf{A}/\Psi$ can be embedded in the ultraproduct $(\prod \mathbf{A}_t)/\mathsf{D}$ which is simple ($\mathcal{V}$ is finitely generated). Since $U \in \mathsf{D}$, $\alpha \not\equiv \beta \pmod{\Psi}$, hence by the construction of α and β, $c/\Psi \neq d/\Psi$ and $\{a_0/\Psi, \ldots, a_{r-1}/\Psi\}$ forms a subalgebra of $\mathbf{A}/\Psi$ isomorphic to $\mathbf{L}$. But for every $x \in A$, $V_x \in \mathsf{D}$ implies that $x \equiv a_i \pmod{\Psi}$ some $i < r$. Therefore $\mathbf{L} \cong \{a_0/\Psi, \ldots, a_{r-1}/\Psi\} = \mathbf{A}/\Psi$ and the Theorem follows.

To verify the claim, it suffices to show that the family $\{U\} \cup \{\, V_x : x \in A \,\}$ has the finite intersection property. For this choose $x_0, x_1, \ldots, x_{k-1}$ from A for some natural number k and let $\mathbf{E}$ be the subalgebra of $\mathbf{A}$ generated by all the elements $a_0, a_1, \ldots, a_{r-1}, b, x_0, x_1, \ldots, x_{k-1}, \alpha$ and β. This is a finite set so, since $\mathcal{V}$ is finitely generated, E is finite. Therefore, by Theorem 2(2), $\mathbf{E}$ is a direct product of simple algebras, in fact $\mathbf{E} \cong \prod(\mathbf{E}_t : t \in T_0)$ where T_0 is a finite subset of T and $\mathbf{E}_t = \mathbf{E}/(\Theta_t\restriction_E)$.

Now suppose $\{U\} \cap \bigcap(V_{x_j} : j < k) = \emptyset$. Then for every $t \in U$, there is an integer $t^* < k$ with $t \notin V_{x_{t^*}}$. Since $\mathbf{E}$ is a direct product, there is an element $e \in E$ such that for every $t \in T_0 \cap U$, $e \equiv x_{t^*} \pmod{\Theta_t}$. Recall that $\mathbf{A}$ is assumed to satisfy $\mathrm{Sep}_{\mathbf{L}}(c, d)$ with $a_0, a_1, \ldots, a_{r-1}, b$ as witnesses. Since $e \in A$, this ensures that for some $i < r$, $\sigma^{\mathbf{A}}(a_i, e, \alpha) \neq \sigma^{\mathbf{A}}(a_i, e, \beta)$. Since every a_i, e, α and β is a member of E this can be computed in $\mathbf{E}$ as well, thus $\sigma^{\mathbf{E}}(a_i, e, \alpha) \neq \sigma^{\mathbf{E}}(a_i, e, \beta)$. However terms of $\mathbf{E}$ are computed coordinatewise, so for any $i < r$, if $t \in T_0 \cap U$ then $e \equiv x_{t^*} \not\equiv a_i \pmod{\Theta_t}$ (since $t \notin V_{x_{t^*}}$), and for $t \in T_0 - U$, $\alpha \equiv \beta \pmod{\Theta_t}$. Therefore the elements $\sigma^{\mathbf{E}}(a_i, e, \alpha)$ and $\sigma^{\mathbf{e}}(a_i, e, \beta)$ agree in every coordinate of $\mathbf{E}$, so must be equal. This is a contradiction and concludes the proof. $\square$

Let us now turn to the amalgamation class. $\mathrm{Amal}(\mathcal{V})$ has proved to be a difficult class to characterize, even for very well-behaved varieties. The aim of this paper is to show that, at least for discriminator varieties, the class has a very satisfactory description, namely by a set of first order sentences. [1] is an in-depth study of the subject and contains the characterization of $\mathrm{Amal}(\mathcal{V})$ that will serve as the starting point here.

Definition 5. Let $\mathcal{V}$ be a variety, $\mathbf{A} \in \mathcal{V}$. We define

(1) $\mathcal{V}_{\mathrm{asi}} = \mathrm{Amal}(\mathcal{V}) \cap \mathcal{V}_{\mathrm{si}}$;
(2) $\mathbf{A}^{\$} = \prod(\mathbf{A}/\Psi : \Psi \in \mathrm{Con}\,\mathbf{A}$ and $\mathbf{A}/\Psi \in \mathcal{V}_{\mathrm{asi}})$;
(3) $\mu_{\mathbf{A}}$ is the canonical homomorphism from $\mathbf{A}$ to $\mathbf{A}^{\$}$.

Werner proved [7, Theorem 2.2(11)] that every discriminator variety is filtral, hence has the congruence extension property. A *maximal simple algebra of* $\mathcal{V}$ is a simple algebra of $\mathcal{V}$ with no proper, simple extensions in $\mathcal{V}$.

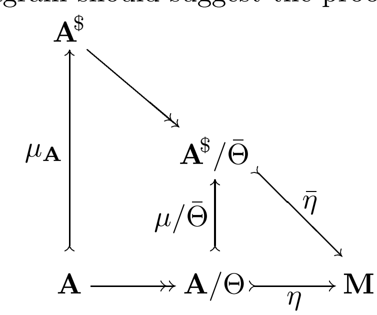
Theorem 6. [1, 3.5 and 4.9] *Let $\mathcal{V}$ be a finitely generated discriminator variety.*

(1) *For $\mathbf{A} \in \mathcal{V}_{\mathrm{si}}$, $\mathbf{A} \in \mathcal{V}_{\mathrm{asi}}$ if and only if for every pair of maximal simple algebras $\mathbf{B}_0, \mathbf{B}_1$ extending $\mathbf{A}$, there is an isomorphism of $\mathbf{B}_0$ with $\mathbf{B}_1$ which is the identity on A.*
(2) *For $\mathbf{A} \in \mathcal{V}$, $\mathbf{A} \in \mathrm{Amal}(\mathcal{V})$ if and only if for every maximal simple algebra $\mathbf{M}$ and homomorphism $\lambda\colon \mathbf{A} \to \mathbf{M}$, there is a homomorphism $\bar{\lambda}\colon \mathbf{A}^{\$} \to \mathbf{M}$ such that $\bar{\lambda} \circ \mu_{\mathbf{A}} = \lambda$.*

Corollary 7. *Let $\mathcal{V}$ be a finitely generated discriminator variety, $\mathbf{A} \in \mathcal{V}$. Then $\mathbf{A} \in \mathrm{Amal}(\mathcal{V})$ if and only if*

(i) *$\mu_{\mathbf{A}}$ is one-to-one, and*
(ii) *for every maximal simple $\mathbf{M}$, every $\Theta \in \mathrm{Con}(\mathbf{A})$ and η embedding $\mathbf{A}/\Theta$ into $\mathbf{M}$, there exists $\bar{\Theta} \in \mathrm{Con}(\mathbf{A}^{\$})$ and $\bar{\eta}$ embedding $\mathbf{A}^{\$}/\bar{\Theta}$ into $\mathbf{M}$ such that $\bar{\Theta}\restriction_A = \Theta$ and $\bar{\eta} \circ (\mu/\bar{\Theta}) = \eta$. (Here $\mu/\bar{\Theta}\colon \mathbf{A}/\Theta \to \mathbf{A}^{\$}/\bar{\Theta}$ takes a/Θ to $\mu(a)/\bar{\Theta}$.)*

Proof. The following diagram should suggest the proof with $\Theta = \ker \lambda$.



□

Suppose $\mathcal{V}$ is of finite type, $\mathbf{K}$, $\mathbf{L}$ are simple algebras of $\mathcal{V}$ and ν is an embedding of $\mathbf{K}$ into $\mathbf{L}$. Write $K = \{k_0, k_1, \ldots, k_{r-1}\}$. By an argument almost identical to the one preceding Theorem 4, there are terms γ and δ so that the formula $\mathrm{Fac}_{\mathbf{K}}(x_0, x_1, \ldots, x_{r-1}, u, v)$ given by

$$(\exists y)(\forall z)\ \Big[\gamma(\vec{x}, y, u, v) \not\approx \delta(\vec{x}, y, u, v) \wedge \bigvee_{i<r} \big(\sigma[x_i, z, \gamma(\vec{x}, y, u, v)] \not\approx \sigma[x_i, z, \delta(\vec{x}, y, u, v)]\big)\Big]$$

is such that

$\mathbf{A} \vDash \mathrm{Fac}_{\mathbf{K}}(\vec{a}, c, d)$ if and only if there is a coatom Ψ of $\mathrm{Con}(\mathbf{A})$ such that $c \equiv d \pmod{\Psi}$ and $\{a_0/\Psi, \ldots, a_{r-1}/\Psi\} \cong \mathbf{K}$ with a_j/Ψ being mapped to k_j for all $j < r$.

Similarly, there is a formula $\mathrm{Ext}_{\mathbf{L},\nu,\mathbf{K}}(\vec{x},\vec{y},u,v)$ such that

> $\mathbf{A} \vDash \mathrm{Ext}_{\mathbf{L},\nu,\mathbf{K}}(\vec{a},\vec{b},c,d)$ if and only if there is a coatom Ψ of $\mathrm{Con}(\mathbf{A})$ such that $c \equiv d \pmod{\Psi}$, $A/\Psi = \{a_0/\Psi,\dots,a_{r-1}/\Psi, b_0/\Psi,\dots,b_{s-1}/\Psi\}$ and there is an isomorphism of $\mathbf{A}/\Psi$ with $\mathbf{L}$ which carries a_j/Ψ to $\nu(k_j)$ for all $j < r$.

Theorem 8. *Let $\mathcal{V}$ be a finitely generated discriminator variety of finite type. Then* $\mathrm{Amal}(\mathcal{V})$ *is finitely axiomatizable.*

Proof. Begin with the set Σ' axiomatizing $\boldsymbol{Ps}(\mathcal{V}_{\mathrm{asi}})$ provided by Theorem 5. Then $\mathbf{A} \vDash \Sigma'$ if and only if $\mu_{\mathbf{A}}$ is one-to-one. To apply Corollary 7, let $\mathbf{M}$ be maximal simple, $\Theta \in \mathrm{Con}(\mathbf{A})$ and η an embedding of $\mathbf{A}/\Theta$ into $\mathbf{M}$. Since $\mathcal{V}$ has the congruence extension property, $\mathbf{K} = \mathbf{A}/\Theta$ is simple. We need a sentence equivalent to the existence (in the presence of Σ') of $\bar{\eta}$ and $\bar{\Theta}$ as in 7(ii).

Since $\mathcal{V}$ is finitely generated, there are only finitely many pairs (up to isomorphism) $\langle \mathbf{L}_i, \nu_i\rangle$ such that: (i) $\mathbf{L}_i \in \mathcal{V}_{\mathrm{asi}}$, (ii) ν_i is an embedding of $\mathbf{K}$ into $\mathbf{L}_i$ and (iii) there exists $\tau\colon \mathbf{L}_i \to \mathbf{M}$ such that $\tau \circ \nu_i = \eta$. Let $i = 0, 1, \dots, m-1$ enumerate those pairs and let $\mathrm{P}_{\mathbf{K},\eta}$ be the sentence

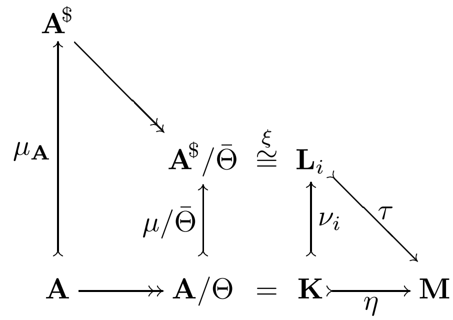
$$(\forall x) \bigvee_{i=0}^{m-1} (\forall u, v)\ \big[\mathrm{Fac}_{\mathbf{K}}(\vec{x},u,v) \to (\exists \vec{y})\ \mathrm{Ext}_{\mathbf{L}_i,\nu_i,\mathbf{K}}(\vec{x},\vec{y},u,v)\big].$$

We verify that if $\mathbf{A} \vDash \Sigma' \cup \{\mathrm{P}_{\mathbf{K},\eta}\}$ then $\bar{\Theta}$ and $\bar{\eta}$ exist with the desired properties. The converse is left to the reader. Choose a sequence $a_0, a_1, \dots, a_{r-1}$ of coset representatives for A by Θ. Since $\mathbf{A}/\Theta = \mathbf{K}$, for any $(c,d) \in \Theta$ we have $\mathbf{A} \vDash \mathrm{Fac}_{\mathbf{K}}(\vec{a},c,d)$. Therefore, by assumption there is an $i < m$ such that $\mathbf{A} \vDash (\exists \vec{y})\ \mathrm{Ext}_{\mathbf{L}_i,\nu_i,\mathbf{K}}(\vec{a},\vec{y},c,d)$, whenever $(c,d) \in \Theta$.

Let $T = \{\, \Psi \in \mathrm{Con}(\mathbf{A}) : \mathbf{A}/\Psi \in \mathcal{V}_{\mathrm{asi}} \,\}$. Let U consist of those $\Psi \in T$ such that there is an isomorphism of $\mathbf{A}/\Psi$ with $\mathbf{L}_i$ taking a_j/Ψ to $\nu_i(k_j)$, for all $j < r$. Also, for each pair (c,d) in Ψ, let $V(c,d) = \{\, \Psi \in T : (c,d) \in \Psi \,\}$. Consider the family $\mathsf{F} = \{U\} \cup \{\, V(c,d) : (c,d) \in \Theta \,\}$. We claim that F is contained in an ultrafilter over T. To show this, it suffices to check the finite intersection property. So, let $p < \omega$ and $(c_0,d_0), \dots, (c_p,d_p)$ be pairs from Θ. In a discriminator variety, every compact congruence is principal (see [7, 2.2.(8)]) so there exists $c, d \in A$ such that for every $\Psi \in \mathrm{Con}(\mathbf{A})$, $(c,d) \in \Psi$ if and only if $(c_j,d_j) \in \Psi$, for all $j \le p$. In particular, $(c,d) \in \Theta$ so $\mathbf{A} \vDash (\exists \vec{y})\ \mathrm{Ext}_{\mathbf{L}_i,\nu_i,\mathbf{K}}(\vec{a},\vec{y},c,d)$. By the very construction of the formula Ext, there is a congruence Ψ contained in $\{U\} \cap V(c,d)$ and hence in each $V(c_j,d_j)$, $j \le p$. Since p as well as the (c_j,d_j)s were arbitrary, F has the finite intersection property.

Thus there is an ultrafilter D over T containing these sets. Set $\bar{\Theta} = \mathsf{D}$ as a congruence on $\mathbf{A}^{\$}$. Since every $V(c,d) \in \mathsf{D}$, $\bar{\Theta}{\restriction}_A \supseteq \Theta$. Since $U \in \mathsf{D}$, we get the opposite inclusion as well as an isomorphism ξ of $\mathbf{A}^{\$}/\bar{\Theta}$ with $\mathbf{L}_i$ whose composition with $\mu/\bar{\Theta}$ equals ν_i. Setting $\bar{\eta}$ to be $\tau \circ \xi$ (the map associated with $(\mathbf{L}_i,\nu_i)$) one verifies that $\bar{\eta} \circ (\mu/\bar{\Theta}) = \eta$. (See diagram.)

Finally, the proof can be completed by observing that there are only finitely many pairs $(\mathbf{K},\eta)$ (up to isomorphism) such that η is an embedding of $\mathbf{K}$ into a maximal simple algebra. Define P to be the formula $\bigwedge \mathrm{P}_{\mathbf{K},\eta}$, the conjunction over all such pairs. Then the set $\Sigma' \cup \{\mathrm{P}\}$ axiomatizes $\mathrm{Amal}(\mathcal{V})$. $\square$



A. careful examination of the sentences involved will reveal that the characterizations in Theorems 5 and 8 are $\forall\exists\forall$ in complexity. It is not hard to show that for any variety $\mathcal{V}$, Amal($\mathcal{V}$) is closed under unions of chains (take an ultraproduct). Thus, by the "Chang-Łos-Susko theorem", there is an axiomatization which is $\forall\exists$ in complexity. This can be achieved by omitting the subformulas $(\forall z)\ \bigvee\big[\sigma(x,z,\alpha) \not\approx \sigma(x,z,\beta)\big]$ from Fac, Ext, and Sep. Since the proofs are more complicated, we have not taken that tack. Is a similar reduction possible for $\boldsymbol{Ps}(\mathcal{S})$?

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