CATEGORICAL EQUIVALENCE OF MODES

CLIFFORD BERGMAN AND JOEL BERMAN

Abstract. A description of all varieties categorically equivalent to a fixed finitely generated variety of modes is provided. The characterization is in terms of a set of basic operations symbols and identities.

For the special case of semilattices, a one-to-one correspondence between the categorically equivalent varieties and the isomorphism types of finite posets is exhibited. Applications to rectangular and left-normal bands, and to affine modules are also discussed.

Two varieties of algebras are generally considered to be "the same" if they are term-equivalent. Roughly speaking, this means that the set of available term-operations in each variety is the same. It is easy to see that a variety that is term-equivalent to a variety of modes is itself a variety of modes.

However, there is a coarser notion of equivalence that may also be of interest. Since varieties can be considered categories, with the algebras as objects and the homomorphisms as the arrows, we might ask whether two particular varieties form equivalent categories.

In this paper we provide a characterization of those varieties (up to term-equivalence) that are categorically equivalent to a fixed finitely generated variety, $\mathcal{V}$, of modes. We do this by expanding the set of basic operation symbols and adding new identities to an equational base for $\mathcal{V}$.

At the end of Section 1 we consider several familiar varieties of modes, such as left-normal bands, rectangular bands and affine modules, and use our technique to construct sample varieties equivalent to each. In Section 2 we look at semilattices in detail, and construct a bijection between all varieties that are categorically equivalent to the variety of semilattices (up to term-equivalence) and all finite posets (up to isomorphism). Again, several examples are presented.

1. Modes

Our universal algebraic notation and terminology largely follows that of [MMT87]. In particular, the definition of a *clone,* of *term-equivalence,* and of *weak isomorphism* can be found in that text. We write $\mathbf{A} \equiv_w \mathbf{B}$ to indicate that the algebras $\mathbf{A}$ and $\mathbf{B}$ are weakly isomorphic. Similarly, we use '$\equiv_t$' to indicate term-equivalence of either algebras or varieties.

Date: December 19, 2000.

1991 *Mathematics Subject Classification.* Primary 18C05 08B99; Secondary 06A12.

Key words and phrases. categorical equivalence, diagonal operation, entropic, idempotent operation, Kronecker product, matrix power, mode, semilattice, variety.

Definition 1.1. Let $\mathbf{A} = \langle A, F \rangle$ be an algebra.

- $\mathbf{A}$ is *idempotent* if for every $f \in F$ and $a \in A$, $f(a, a, \dots, a) = a$.
- $\mathbf{A}$ is *entropic* if for every $f, g \in F$ and every $a_{ij} \in A$, (for $1 \leq i \leq n$, $1 \leq j \leq k$),

$$f\big(g(a_{11}, \dots, a_{1k}), \dots, g(a_{n1}, \dots, a_{nk})\big) = \\ g\big(f(a_{11}, \dots, a_{n1}), \dots, f(a_{1k}, \dots, a_{nk})\big).$$

  Here, f is an n-ary, and g a k-ary operation on A. We often say that the operations f and g *commute* when this relationship obtains.
- An idempotent, entropic algebra is called a *mode.*
- A variety of algebras is idempotent, entropic, or a mode variety, if and only if every member has the corresponding property.

There are several equivalent formulations of the above conditions. $\mathbf{A}$ is idempotent if and only if every one-element subset of A is a subalgebra. $\mathbf{A}$ is entropic if and only if every basic operation (say of rank n) is a homomorphism from $\mathbf{A}^n$ to $\mathbf{A}$. And as Keith Kearnes observed, $\mathbf{A}$ is a mode if and only if every *polynomial* operation of $\mathbf{A}$ is such a homomorphism, see [Kea].

Any variety can be viewed as a category, in which the algebras are the objects, and the homomorphisms are the arrows. If $\mathcal{V}$ and $\mathcal{W}$ are varieties of algebras, then $\mathcal{V}$ is categorically equivalent to $\mathcal{W}$ (in symbols, $\mathcal{V} \equiv_c \mathcal{W}$) if there are functors $F \colon \mathcal{V} \to \mathcal{W}$ and $G \colon \mathcal{W} \to \mathcal{V}$ such that the composite functors FG and GF are naturally isomorphic to the identity functors on the respective varieties. The functors F and G can each be referred to as an equivalence of categories. See [Mac71] for the formal definitions of these concepts.

It is often convenient to localize the notion of categorical equivalence to individual algebras. Thus for algebras $\mathbf{A}$ and $\mathbf{B}$, we write $\mathbf{A} \equiv_c \mathbf{B}$ if there is an equivalence of categories $F \colon \mathbf{V}(\mathbf{A}) \to \mathbf{V}(\mathbf{B})$ such that $F(\mathbf{A}) = \mathbf{B}$. Here, the variety generated by an algebra $\mathbf{A}$ is denoted $\mathbf{V}(\mathbf{A})$.

The purpose of this paper is to explore the behavior of idempotence and entropicity under categorical equivalence. Certainly, neither property is preserved by categorical equivalence. In fact, almost the exact opposite is true. For example, if $\mathcal{V} \equiv_c \mathcal{W}$ and if $\mathcal{V}$ and $\mathcal{W}$ are both idempotent or both entropic, then $\mathcal{V} \equiv_t \mathcal{W}$. See [Lež82] for the idempotent case and [Wra70, Section 15] for entropicity. Given a finite mode $\mathbf{A}$, Theorems 1.9 and 1.10 provide a reasonably concise characterization of all varieties categorically equivalent to $\mathbf{V}(\mathbf{A})$ and of all algebras categorically equivalent to $\mathbf{A}$. In the special case that $\mathbf{A}$ is the two-element semilattice, we are able to further improve these results.

The crucial ingredient in this is McKenzie's characterization of categorical equivalence, [McK94]. We summarize the main ideas.

Definition 1.2. Let $\mathbf{A}$ be an algebra of similarity type $\mathcal{I}$, n a positive integer, and s a unary term operation of $\mathbf{A}$. For any $k \in \omega$, let I_k denote the set of k-ary terms of the similarity type of $\mathcal{I}$.

- For every positive integer p and every sequence $g_1, g_2, \ldots, g_n$ of pn-ary operations on A, $(g_1, \ldots, g_n)$ denotes the p-ary operation on A^n that maps $(\overline{a}_1, \ldots, \overline{a}_p)$ to $(\, g_1(\tilde{\mathbf{a}}), g_2(\tilde{\mathbf{a}}), \ldots, g_n(\tilde{\mathbf{a}})\,)$, where $\overline{a}_i = (a_{1i}, \ldots, a_{ni})$ is an element of A^n, and
$$\tilde{\mathbf{a}} = (a_{11}, a_{21}, \ldots, a_{n1}, a_{12}, \ldots, a_{np}) \in A^{pn}.$$
- The *n-th matrix power* of $\mathbf{A}$ is the algebra $\mathbf{A}^{[n]}$ with universe A^n and similarity type indexed by $\bigcup_{p\in\omega}(I_{np})^n$. For each n-tuple of terms $(g_1, \ldots, g_n)$, $\mathbf{A}^{[n]}$ has a basic np-ary operation as described above.
- The unary operation s is *idempotent* if for every $x \in A$, $s\big(s(x)\big) = s(x)$, and s is *invertible* if for some r there are $w \in \mathrm{Clo}_r(\mathbf{A})$ and $t_1, \ldots, t_r \in \mathrm{Clo}_1(\mathbf{A})$ such that, for every $a \in A$, $w\big(st_1(a), st_2(a), \ldots, st_r(a)\big) = a$.
- Let s be an idempotent term operation of $\mathbf{A}$. By $\mathbf{A}(s)$ we denote the algebra with universe $s(A)$ and similarity type indexed by $\bigcup_{p\in\omega} I_p$. For each natural number p and term $g \in I_p$, $\mathbf{A}(s)$ has a fundamental operation $g_s = s \circ g\restriction_{s(A)}$.

It is unfortunate that the term "idempotent" should occur in two different roles in this paper. However, both uses are quite standard in the literature. Note that the only unary operation that satisfies the idempotence condition of Definition 1.1 is the identity map. So we adopt the following convention: When idempotence is used in reference to a unary operation, it is in the sense of Definition 1.2, whereas, for an operation of rank greater than 1, it refers to Definition 1.1.

Let $\mathcal{V}$ be a variety of algebras. For a positive integer k, $\mathcal{V}^{[k]}$ denotes the class of all algebras isomorphic to $\mathbf{A}^{[k]}$, for some $\mathbf{A} \in \mathcal{V}$. The class $\mathcal{V}^{[k]}$ is again a variety, see [Tay75, 0.13] for a proof. For any invertible idempotent term s of $\mathcal{V}$, $\mathcal{V}(s)$ denotes the class $\{\, \mathbf{A}(s) : \mathbf{A} \in \mathcal{V}\,\}$. This class is also a variety, see [McK94, Theorem 2.1].

McKenzie's Theorem. Two varieties, $\mathcal{V}$ and $\mathcal{W}$, are categorically equivalent if and only if there exists a positive integer m and unary term s on $\mathcal{V}^{[m]}$ such that $\mathcal{V}^{[m]}(s) \equiv_t \mathcal{W}$, with s invertible and idempotent. For two algebras $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \equiv_c \mathbf{B}$ if and only if there exist a positive integer m and unary term s such that $\mathbf{B}$ is term-equivalent to an algebra isomorphic to $\mathbf{A}^{[m]}(s)$, with s invertible and idempotent.

When $\mathbf{A} \equiv_c \mathbf{B}$, the functor $\mathbf{C} \mapsto \mathbf{C}^{[m]}(s)$ realizes (up to term-equivalence) a categorical equivalence between $\mathsf{V}(\mathbf{A})$ and $\mathsf{V}(\mathbf{B})$.

One could say that McKenzie's Theorem provides a complete solution to the problems we have set for ourselves. However, the algebra $\mathbf{A}^{[m]}(s)$ is, by definition, an untyped object. That is, rather than being presented as a set together with a reasonably small, perhaps finite, but at least manageable sequence of basic operations, we get the entire clone of term operations in no particular order. McKenzie does show in [McK94, Theorem 6.8] that if $\mathcal{V}$ is of finite type (or finitely based), then so is $\mathcal{V}^{[m]}(s)$. Our results for modes are an improvement (in the number and rank of the operations involved, and the complexity of the identities) on what would be obtained from McKenzie's construction.

Furthermore, given an algebra $\mathbf{A}$ it is not obvious how to find the invertible idempotent terms on $\mathbf{A}^{[m]}$. This is still not easy for an arbitrary mode. However, our first result shows that for modes, we get invertibility for free.

Definition 1.3. Let A be a set, and k a positive integer. We define a unary operation $\tilde{t}_i$ (for each $1 \leq i \leq k$), and a k-ary operation $\tilde{d}$ on A^k by

$$\tilde{d}\big((x_{11}, x_{12}, \dots, x_{1k}), \dots, (x_{k1}, x_{k2}, \dots x_{kk})\big) = (x_{11}, x_{22}, \dots . x_{kk})$$
$$\tilde{t}_i(x_1, x_2, \dots, x_k) = (x_i, x_i, \dots, x_i), \qquad \text{for } i = 1, \dots, k.$$

Also, for each r-ary operation f on A, we have an r-ary operation $f^{[k]}$ on A^k given by

$$f^{[k]}\big((x_{11}, x_{12}, \dots, x_{1k}), \dots, (x_{r1}, \dots, x_{rk})\big) = \big(f(x_{11}, x_{21}, \dots, x_{r1}), \dots, f(x_{1k}, \dots, x_{rk})\big)$$

The operation $f^{[k]}$ is nothing but the usual component-wise application of f.

Lemma 1.4. *Let $\mathcal{V}$ be any variety, $k \geq 1$ and $s = (s_1, \dots, s_k)$ a unary term on $\mathcal{V}^{[k]}$. Suppose that for some $i \leq k$ and unary term u of $\mathcal{V}$, the identity $u(s_i(x, x, \dots, x)) = x$ holds in $\mathcal{V}$. Then s is invertible on $\mathcal{V}^{[k]}$.*

Proof. For simplicity, let us assume that $i = 1$. Let $\mathbf{A} \in \mathcal{V}$ and $x_1, \dots, x_k$ be members of A. Consider the k-ary term f on $\mathcal{V}^{[k]}$ given by

$$f\big((x_{11}, \dots, x_{k1}),\, (x_{12}, \dots, x_{k2}), \dots, (x_{1k}, \dots, x_{kk})\big) = \big(u(x_{11}), u(x_{12}), \dots, u(x_{1k})\big).$$

Writing $\bar{x} = (x_1, x_2, \dots, x_k)$ note that for any $i \leq k$,

$$s\tilde{t}_i(\bar{x}) = \big(s_1(x_i, \dots, x_i),\, s_2(x_i, \dots, x_i), \dots, s_k(x_i, \dots, x_i)\big).$$

Therefore, we compute

$$f\big(s\tilde{t}_1(\bar{x}), s\tilde{t}_2(\bar{x}), \dots, s\tilde{t}_k(\bar{x})\big) = \big(us_1(x_1, \dots, x_1),\, us_1(x_2, \dots, x_2), \dots, us_1(x_k, \dots, x_k)\big) = \bar{x}.$$

Thus, s is invertible. □

Corollary 1.5. *Let $\mathcal{V}$ be a variety in which the free $\mathcal{V}$-algebra on one generator has no proper subalgebras. Then every unary term on $\mathcal{V}^{[k]}$ is invertible. In particular, every idempotent variety, hence every variety of modes, has this property.*

Proof. Let $s = (s_1, \dots, s_k)$ be a unary term on $\mathcal{V}^{[k]}$, and $v(x) = s_1(x, \dots, x)$. We can think of $v(x)$ as an element of $\mathbf{F}$, the free algebra generated by x. Then by assumption, x is a member of the subalgebra of $\mathbf{F}$ generated by $v(x)$. Therefore, for some unary term $u(x)$ on $\mathcal{V}$, we have $x = u(v(x)) = u(s_1(x, x, \dots, x))$. The result now follows from Lemma 1.4. □

Lemma 1.6. *Let A be a finite set and C, D two clones on A. Suppose that $C \subseteq D$ and that the algebras $\langle A, C \rangle$ and $\langle A, D \rangle$ are categorically equivalent. Then $C = D$.*

Proof. Let us write $\mathrm{Sub}(\langle A, C\rangle^k)$ for the set of subalgebras of the algebra $\langle A, C\rangle^k$. It is well-known that C is precisely the set of operations on A that preserve all members of $\mathrm{Sub}(\langle A, C\rangle^k)$, for all positive integers k.

Suppose that $C \subsetneq D$. Then for some k, $\mathrm{Sub}(\langle A, C \rangle^k) \supsetneq \mathrm{Sub}(\langle A, D \rangle^k)$. However, the categorical equivalence implies that for every k, the lattices $\mathrm{Sub}(\langle A, C \rangle^k)$ and $\mathrm{Sub}(\langle A, D \rangle^k)$ are isomorphic, hence have the same cardinality. Since A is finite, this is impossible. □

Our objective is to give a description (up to term-equivalence) of those varieties that are categorically equivalent to a fixed, finitely generated variety of modes. These varieties are described in the following definition.

Definition 1.7. Let $\mathcal{M}$ be a finitely generated variety of modes. Let the set of basic operation symbols be F. For each $f \in F$, the rank of f will be denoted $r(f)$. Fix an equational base, Σ, for $\mathcal{M}$. Let k be a positive integer and $s = (s_1, s_2, \dots, s_k)$ an invertible idempotent term on $\mathcal{M}^{[k]}$.

We define the (k, s)-*expansion of* $\mathcal{M}$, denoted $E(\mathcal{M}, k, s)$, to be the following variety. The similarity type of $E(\mathcal{M}, k, s)$ is $F \cup \{d, t_1, t_2, \dots, t_k\}$, in which d, $t_1, \dots, t_k$ are operation symbols not appearing in F. The rank of d is k, the rank of each t_i is 1. The identities defining $E(\mathcal{M}, k, s)$ are

$$
\begin{array}{lll}
 & \mathrm{ax1} & \text{all identities of } \Sigma; \\
 & \mathrm{ax2}_{ij} & t_i t_j(x) = t_j(x); \\
(1) & \mathrm{ax3} & d\big(t_1(x), t_2(x), \dots, t_k(x)\big) = x; \\
 & \mathrm{ax4}_{i,f} & t_i\big(f\big(x_1, x_2, \dots, x_{r(f)}\big)\big) = f\big(t_i(x_1), \dots, t_i\big(x_{r(f)}\big)\big); \\
 & \mathrm{ax5}_i & t_i d(x_1, x_2, \dots, x_k) = s_i\big(t_1(x_1), \dots, t_k(x_k)\big).
\end{array}
$$

In each axiom schema, i and j run from 1 to k, and f ranges over F.

Let us make several remarks on this definition. First, if $k > 1$ then it is possible to derive the axioms $\mathrm{ax2}_{ii}$ from the remaining ones. For, if $i \neq j$, then $t_i = t_j t_i = t_i t_j t_i = t_i t_i$. We use this fact without comment in several of the examples. Second, for any $\mathbf{A} \in E(\mathcal{M}, k, s)$, it follows from ax2 that $t_1(A) = t_2(A) = \cdots = t_k(A)$, and from ax3 that the mapping $a \mapsto (t_1(a), \dots, t_k(a))$ is injective. Finally, note that the similarity type of $E(\mathcal{M}, k, s)$ is an expansion of that of $\mathcal{M}$. Since s_i is a term of $\mathcal{M}$, it can also be viewed as a term of $E(\mathcal{M}, k, s)$

Lemma 1.8. *Let $\mathcal{M}$ be a variety of modes, and let $\mathcal{V} = E(\mathcal{M}, k, s)$ for some k and idempotent term s. Then t_1 is an invertible idempotent term for $\mathcal{V}$, and the variety $\mathcal{V}(t_1)$ is term-equivalent to a subvariety, $\mathcal{M}'$, of $\mathcal{M}$.*

Proof. That every t_i is an invertible idempotent follows from ax2 and ax3. For the rest of the proof, we shall write t instead of t_1 to save a subscript. Let $\mathbf{B}$ be an algebra in $\mathcal{V}$. Thus $\mathbf{B}(t)$ is a typical member of $\mathcal{V}(t)$. Recall from Definition 1.2 that for any term g of $\mathcal{M}$, the term operation g_t on $\mathbf{B}(t)$ is defined to be $t \circ g\restriction_{t(B)}$. Let $F_t = \{\, f_t : f \in F \,\}$. These are among the term-operations of $\mathbf{B}(t)$. Set $T = t(B)$. From ax1 and ax4 we deduce that the algebra $\langle\, T, F_t \,\rangle$ satisfies Σ, hence is a member of $\mathcal{M}$.

Claim. *For every $g \in \operatorname{Clo}(\mathbf{B})$ and every $1 \leq i \leq k$, the operation g_{t_i} lies in the clone on T generated by F_t.*

Proof. Let C denote the clone on T generated by F_t. Using ax2 and ax4, it is easy to see that for every F-term p, the operaton p_t is a member of C. We prove the claim by induction on the height of g. For the base case, suppose that $g(x) = x$. Let $y \in T$. Then $y = t(x)$ for some $x \in B$. Therefore $g_{t_i}(y) = t_i g(y) = t_i(y) = t_i t(x) = t(x) = y$, by ax2. Thus g_{t_i} is a projection operation on T.

Now suppose that $g(x_1, x_2, \dots, x_n) = t_j h(x_1, \dots, x_n)$ for some $j \leq k$ and $h \in \operatorname{Clo}(\mathbf{B})$. Then, for all $\bar{y} = y_1, \dots, y_n \in T$, $g_{t_i}(\bar{y}) = t_i g(\bar{y}) = t_i t_j h(\bar{y}) = t_j h(\bar{y}) = h_{t_j}(\bar{y})$. By the induction hypotheses, $h_{t_j} \in C$. Similarly, if $g(\bar{x}) = f(h_1(\bar{x}), h_2(\bar{x}), \dots, h_r(\bar{x}))$ for some $f \in F$, then (using ax4),

$$\begin{aligned} g_{t_i}(\bar{y}) = t_i g(\bar{y}) &= t_i f\big(h_1(\bar{y}), \dots, h_r(\bar{y})\big) = t t_i f\big(h_1(\bar{y}), \dots, h_r(\bar{y})\big) \\ &= t f\big(t_i h_1(\bar{y}), \dots, t_i h_r(\bar{y})\big) = f_t((h_1)_{t_i}(\bar{y}), , \dots, (h_r)_{t_i}(\bar{y})) \in C. \end{aligned}$$

Finally, suppose $g(\bar{x}) = d\big(h_1(\bar{x}), \dots, h_k(\bar{x})\big)$. Then for $\bar{y} \in T$, by ax5,

$$\begin{aligned} g_{t_i}(\bar{y}) &= t_i d\big(h_1(\bar{y}), \dots, h_k(\bar{y})\big) = t t_i d\big(h_1(\bar{y}), \dots, h_k(\bar{y})\big) \\ &= t s_i\big(t_1 h_1(\bar{y}), \dots, t_k h_k(\bar{y})\big) = (s_i)_t\big(\, (h_1)_{t_1}(\bar{y}), \dots, (h_k)_{t_k}(\bar{y}) \,\big). \end{aligned}$$

Since s_i is an F-term, we have $(s_i)_t \in C$. Also, every $(h_i)_{t_i} \in C$. Consequently, g_{t_i} lies in C. □

From the claim, it follows that $\operatorname{Clo}(\mathbf{B}(t))$ is a subset of the clone on T generated by F. But since $F_t \subseteq \operatorname{Clo}(\mathbf{B}(t))$, the algebras $\mathbf{B}(t)$ and $\langle\, T, F_t \,\rangle$ are term-equivalent. Therefore, $\mathcal{V}(t)$ is term-equivalent to its image under the functor that sends $\mathbf{B}(t)$ to $\langle\, t(B), F_t \,\rangle$. □

Let $\mathbf{M}$ be a finite generator of $\mathcal{M}$. Among the term-operations of $\mathbf{M}^{[k]}$ are all operations $f^{[k]}$ for $f \in F$ (see Definition 1.3). Define

$$\widehat{F} = \{\, f^{[k]} : f \in F \,\} \text{ and}$$
$$D_k(M) = \big\{\, (x, x, \dots, x) \in M^k : x \in M \,\big\}.$$

Obviously $\mathbf{M} \cong \langle D_k(M), \widehat{F} \rangle$.

Now recall that the component functions, $s_1, \ldots, s_k$, of our invertible idempotent term s are k-ary terms of $\mathcal{M}$. Hence, they are idempotent. It follows that the elements of $D_k(M)$ are fixed by s, and by every $\tilde{t}_i$. Consequently, for every $f \in F$ and $\bar{x} \in D_k(M)$, $f^{[k]}_{\tilde{t}_i s}(\bar{x}) = \tilde{t}_i s f^{[k]}(\bar{x}) = f^{[k]}(\bar{x})$. Equivalently, $\mathbf{M} \cong \langle D_k(M), \widehat{F}_s\rangle = \langle D_k(M), \widehat{F}_{\tilde{t}_i s}\rangle$.

Theorem 1.9. *$E(\mathcal{M}, k, s)$ is term-equivalent to $\mathcal{M}^{[k]}(s)$.*

Proof. We shall continue to write $\mathcal{V}$ for $E(\mathcal{M}, k, s)$. Let $\mathbf{A} = \mathbf{M}^{[k]}(s)$ and let $\mathbf{A}' = \langle\, A, \widehat{F}_s, \tilde{d}_s, (\tilde{t}_1)_s, \ldots, (\tilde{t}_k)_s\,\rangle$. Note that $\mathbf{A}'$ is a reduct of $\mathbf{A}$. It is straightforward to check that $\mathbf{A}'$ satisfies all of the identities in (1). (Axiom ax1 follows from the entropicity of $\mathcal{M}$.) Consequently, $\mathbf{A}' \in \mathcal{V}$. Since the members of $D_k(M)$ are fixed by $s\tilde{t}_1 s$, we have $t_1(A') = s\tilde{t}_1(A) = s\tilde{t}_1 s(M^k) = D_k(M)$. By Lemma 1.8, $\mathbf{A}'(t_1) \equiv_t \langle t_1(A'), F_{t_1}\rangle = \langle D_k(M), \widehat{F}_{\tilde{t}_i s}\rangle \cong \mathbf{M}$. It follows that the subvariety $\mathcal{M}'$ constructed in Lemma 1.8 contains $\mathbf{M}$. Since $\mathbf{M}$ generates $\mathcal{M}$, we actually have $\mathcal{M}' = \mathcal{M}$. From this we deduce that $\mathcal{M} \equiv_c \mathcal{V}$. Since the functor that realizes that equivalence carries $\mathbf{M}$ to $\mathbf{A}'$, and $\mathbf{M}$ is generic for $\mathcal{M}$, we conclude that the variety generated by $\mathbf{A}'$ is all of $\mathcal{V}$.

On the other hand, $\mathrm{Clo}(\mathbf{A}') \subseteq \mathrm{Clo}(\mathbf{A})$ and $\mathbf{A}' \equiv_c \mathbf{M} \equiv_c \mathbf{M}^{[k]}(s) = \mathbf{A}$. Therefore, by Lemma 1.6, $\mathrm{Clo}(\mathbf{A}') = \mathrm{Clo}(\mathbf{A})$, i.e., $\mathbf{A}'$ and $\mathbf{A}$ are term-equivalent. Therefore the varieties they generate are term-equivalent as well. □

The algebra $\mathbf{A}'$ that we constructed in Theorem 1.9 turned out to be term-equivalent to $\mathbf{M}^{[k]}(s)$. Therefore, we have the following "local" version of the Theorem.

Theorem 1.10. *Let $\mathbf{M}$ be a finite mode, k a natural number, and s an invertible idempotent term on $\mathbf{M}^{[k]}$. Then $\mathbf{M}^{[k]}(s)$ is term-equivalent to the algebra $\big\langle s(M^k), \widehat{F}_s, \tilde{d}_s, (\tilde{t}_1)_s, \ldots, (\tilde{t}_k)_s\big\rangle \in E(\mathcal{M}, k, s)$.*

There is another way to look at Theorem 1.9. Let $\mathcal{E}$ denote the variety of sets. It is well-known that the variety $\mathcal{E}^{[k]}$ can be presented using basic operations d and c (of ranks k and 1 respectively) and the following identities (see [Tay75]).

$$
\begin{aligned}
&d(x, x, \ldots, x) = x \\
&d\big(d(x_{11}, \ldots, x_{1k}), \ldots, d(x_{k1}, \ldots, x_{kk})\big) = d(x_{11}, \ldots, x_{kk}) \\
&cd(x_1, x_2, \ldots, x_k) = d(c(x_2), \ldots, c(x_k), c(x_1)) \\
&c^k(x) = x.
\end{aligned} \tag{2}
$$

The operation c^k is the composite of c with itself k times.

Now, for $i = 1, \ldots, k$, define $t_i(x) = d(c^{k+i-1}(x), c^{k+i-2}(x), \ldots, c^i(x))$. Then $\mathcal{E}^{[k]}$ can also be presented by operation symbols d, t_1, $\ldots$, t_k and identities

$$(3)\qquad \begin{aligned} &t_i t_j(x) = t_j(x) && i,j = 1,\dots,k\\ &d(t_1(x),\dots,t_k(x)) = x\\ &t_i(d(x_1,\dots,x_k)) = t_i(x_i) && i = 1,\dots,k \end{aligned}$$

This follows from Theorem 1.9 by taking $\mathcal{M} = \mathcal{E}$ and s the identity operation on $\mathcal{E}^{[k]}$. However, it is not difficult to derive (3) directly from (2), and vice-versa.

For an arbitrary variety $\mathcal{W}$, one can axiomatize $\mathcal{W}^{[k]}$ by adding to the description in (3) the operation symbols of $\mathcal{W}$, an equational base $\Sigma_{\mathcal{W}}$ for $\mathcal{W}$, and the axioms

$$(4)\qquad t_i\big(f\big(x_1,\dots,x_{r(f)}\big)\big) = f\big(t_i(x_1),\dots,t_i\big(x_{r(f)}\big)\big) \qquad i = 1,\dots,k$$

as f ranges over all basic operation symbols of $\mathcal{W}$. Note that from the identities in (3)+(4), one can easily derive the identities asserting that d commutes with every basic operation symbol of $\mathcal{W}$.

Now, let us recall that for varieties $\mathcal{V}$ and $\mathcal{W}$, the *Kronecker product,* $\mathcal{V} \otimes \mathcal{W}$, of $\mathcal{V}$ and $\mathcal{W}$, is the variety whose set of basic operation symbols is the disjoint union of those of $\mathcal{V}$ and $\mathcal{W}$, and whose defining equations consist of an equational base for $\mathcal{V}$, an equational base for $\mathcal{W}$, and identities asserting that the basic operations of $\mathcal{V}$ commute with those of $\mathcal{W}$. (See [Fre66, page 93], where the construction was called the tensor product.) From the discussion in the previous paragraphs, we see that the identities (3)+(4)+$\Sigma_{\mathcal{W}}$ describe the variety $\mathcal{E}^{[k]} \otimes \mathcal{W}$. (Thus we have the well-known relationship $\mathcal{W}^{[k]} \equiv_t \mathcal{E}^{[k]} \otimes \mathcal{W}$, for any variety $\mathcal{W}$.)

Finally, let $\mathcal{E}_k$ denote the variety defined only by the identities

$$\begin{aligned} &t_i t_j(x) = t_j(x), \qquad i,j = 1,\dots,k\\ &d(t_1(x),\dots,t_k(x)) = x. \end{aligned}$$

Note that $\mathcal{E}_k$ is a supervariety of $\mathcal{E}^{[k]}$. The identities ax1–ax4 of (1) describe the variety $\mathcal{E}_k \otimes \mathcal{M}$. Thus, Theorem 1.9 asserts that $\mathcal{M}^{[k]}(s)$ is term-equivalent to the subvariety of $\mathcal{E}_k \otimes \mathcal{M}$ defined by ax5.

We are not aware of any appearance of $\mathcal{E}_k$ in the literature. However it is relatively easy to give a description of its members. Just as $\mathcal{E}^{[k]}$ is the theory of (direct) powers of sets, $\mathcal{E}_k$ is a theory of a kind of diagonal subdirect retract of sets.

Let $\langle A, d, t_1, \dots, t_k \rangle$ be a member of $\mathcal{E}_k$. As we noted earlier, there is a subset B of A with $B = t_1(A) = t_2(A) = \cdots = t_k(A)$, and the mapping $x \mapsto \big(t_1(x),\dots,t_k(x)\big)$ is a diagonal subdirect injection of A into B^k. Furthermore, $d{\restriction}_{B^k}$ is a one-sided inverse of this injection.

Conversely, let A and B be sets, $h\colon A \to B^k$ a diagonal subdirect injection and $d'\colon B^k \to A$ such that $d' \circ h$ is the identity on A. By assumption, for each $b \in B$, there is a unique $\bar{b} \in A$ such that $h(\bar{b}) = (b, b, \dots, b)$. Thus we identify B with $\{\, \bar{b} : b \in B \,\}$, and think of B as a subset of A. Let $p_i\colon B^k \to B$ be the i^{th} projection map, for $i = 1, \dots k$, and define $t_i = p_i \circ h$. Since we are treating B as contained in A, t_i is a unary operation on A. Finally, extend d' arbitrarily to a function $d\colon A^k \to A$. It is now a simple matter to verify that $\langle\, A,\, d, t_1, \dots, t_k \,\rangle \in \mathcal{E}_k$.

Let us consider several applications of Theorem 1.9 to some familiar varieties of modes.

Example 1.11. A *left-normal band* is a binar satisfying the identities

$$\begin{array}{ll} \text{(i)} & x \cdot x = x \\ \text{(a)} & x \cdot (y \cdot z) = (x \cdot y) \cdot z \\ \text{(ln)} & x \cdot (y \cdot z) = x \cdot (z \cdot y). \end{array}$$

Every left-normal band is entropic since $(x \cdot y) \cdot (z \cdot w) = (x \cdot (y \cdot z)) \cdot w = (x \cdot (z \cdot y)) \cdot w = (x \cdot z) \cdot (y \cdot w)$. Thus the class of all left-normal bands forms a variety of modes. This variety is generated by the three-element algebra $\mathbf{L}$ with multiplication table given by

$\cdot$	a	b	c
a	a	a	c
b	b	b	c
c	c	c	c

see [PR92, Example 7.3]. We denote this variety by $\mathcal{L}$.

Let us define a unary term s on $\mathcal{L}^{[2]}$ by $s(x, y) = (x \cdot y,\, y)$. That s is idempotent follows from the fact that

$$(x \cdot y) \cdot y = x \cdot (y \cdot y) = x \cdot y \tag{5}$$

which uses only idempotence and associativity. Therefore, by Corollary 1.5, s is invertible, and $\mathcal{L}^{[2]}(s)$ is a variety categorically equivalent to $\mathcal{L}$. We can apply Theorem 1.9 to axiomatize $\mathcal{L}^{[2]}(s)$. We add unary operations t_1 and t_2, and a binary operation '$*$' (instead of the prefix symbol d used in the theorem). An axiom set will consist of the equations (i), (a) and (ln), together with

$$\begin{array}{ll} t_1 t_2(x) = t_2(x), & t_2 t_1(x) = t_1(x) \\ t_1(x) * t_2(x) = x & \\ t_1(x \cdot y) = t_1(x) \cdot t_1(y), & t_2(x \cdot y) = t_2(x) \cdot t_2(y) \\ t_1(x * y) = t_1(x) \cdot t_2(y), & t_2(x * y) = t_2(y). \end{array} \tag{6}$$

The algebra $\mathbf{L}^{[2]}(s)$ will have 7 elements: (a, a), (a, b), (b, a), (b, b), (c, a), (c, b) and (c, c), which we denote by $u_1, u_2, \dots, u_7$. $\mathbf{L}^{[2]}(s)$ is the minimal generator of $\mathcal{L}^{[2]}(s)$. The operation tables for $\mathbf{L}^{[2]}(s)$ are given in Table 1.

To give the reader a better feel for these constructions, we include a few sample computations. For the binary operation '$\cdot$' we have for example

$$u_1 \cdot u_6 = (a, a) \cdot (c, b) = s(a \cdot c,\, a \cdot b) = s(c, a) = (c \cdot a, a) = (c, a) = u_5.$$

$\cdot$	u_1	u_2	u_3	u_4	u_5	u_6	u_7
u_1	u_1	u_1	u_1	u_1	u_5	u_5	u_7
u_2	u_2	u_2	u_2	u_2	u_6	u_6	u_7
u_3	u_3	u_3	u_3	u_3	u_5	u_5	u_7
u_4	u_4	u_4	u_4	u_4	u_6	u_6	u_7
u_5	u_5	u_5	u_5	u_5	u_5	u_5	u_7
u_6	u_6	u_6	u_6	u_6	u_6	u_6	u_7
u_7	u_7	u_7	u_7	u_7	u_7	u_7	u_7

	t_1	t_2
u_1	u_1	u_1
u_2	u_1	u_4
u_3	u_4	u_1
u_4	u_4	u_4
u_5	u_7	u_1
u_6	u_7	u_4
u_7	u_7	u_7

$*$	u_1	u_2	u_3	u_4	u_5	u_6	u_7
u_1	u_1	u_2	u_1	u_2	u_1	u_2	u_7
u_2	u_1	u_2	u_1	u_2	u_1	u_2	u_7
u_3	u_3	u_4	u_3	u_4	u_3	u_4	u_7
u_4	u_3	u_4	u_3	u_4	u_3	u_4	u_7
u_5	u_5	u_6	u_5	u_6	u_5	u_6	u_7
u_6	u_5	u_6	u_5	u_6	u_5	u_6	u_7
u_7	u_5	u_6	u_5	u_6	u_5	u_6	u_7

FIGURE 1. The algebra $\mathbf{L}^{[2]}(s)$

Of course, all of the instances of '$\cdot$' after the first equal sign are computed in the original algebra $\mathbf{L}$. The term $x * y$ is defined to be $s(\tilde{d}(x, y))$. Thus

$$u_2 * u_7 = (a, b) * (c, c) = s(\tilde{d}((a, b),\ (c, c))) = s(a, c) = (a \cdot c,\ c) = (c, c) = u_7.$$

Similarly, $t_1(u_2) = t_1(a, b) = s\tilde{t}_1(a, b) = s(a, a) = (a, a) = u_1$.

Example 1.12. A *rectangular band* is a binar satisfying the equations (i), (a) of Example 1.11 and

$$(r) \qquad x \cdot (y \cdot z) = x \cdot z.$$

It is easy to see that every rectangular band is entropic. Every rectangular band is isomorphic to a product of a left-zero semigroup (i.e., one obeying the law $x \cdot y = x$) and a right-zero semigroup, and conversely. See [MMT87, pg. 117, ex. 10]. Therefore the variety $\mathcal{R}$ of all rectangular bands is generated by a four-element algebra (the direct product of a two-element left-zero semigroup, and a two-element right-zero semigroup). Using equation (5), we see that the unary term $s(x, y) = (x \cdot y,\ y)$ is idempotent, hence invertible, on $\mathcal{R}^{[2]}$.

Using an analysis similar to that of Example 1.11, we arrive at an axiomatization of $\mathcal{R}^{[2]}(s)$ consisting of the same identities as in (6), except that (ln) is replaced by (r).

Example 1.13. Let R be a commutative ring with identity. The variety $\mathcal{R}$ of *affine spaces over* R was discussed earlier in this volume [Smi].

The unary terms on $\mathcal{R}^{[2]}$ are of the form $s(x,y) = (ux + u'y,\, vx + v'y)$, as u and v range through the members of R [Smi, Equation 4.4]. Here, we are using the notational convention $u' = 1 - u$. It is not hard to check that s is idempotent if and only if

$$u'(u - v) = v(u - v) = 0. \tag{7}$$

By Corollary 1.5, s is invertible if and only if s is idempotent.

To take a specific example, let $R = \mathbb{Z}_6$, the ring of integers modulo 6. There are several pairs u, v that satisfy (7): $1, 3$; $1, 4$; $5, 2$. Let us take $s(x,y) = (x,\, 3x-2y)$. We can obtain an axiomatization of the variety $\mathcal{Z}_6^{[2]}(s)$ by combining the axioms (4.10–4.11) of [Smi] with those of our equations (1). Alternately, we can describe $\mathcal{Z}_6^{[2]}(s)$ as the subvariety of $\mathcal{E}_2 \otimes \mathcal{Z}_6$ defined by

$$\begin{aligned} t_1(x * y) &= t_1(x) \\ t_2(x * y) &= 3t_1(x) - 2t_2(y). \end{aligned}$$

There are, of course, numerous other examples that one could develop here. Several examples involving semilattices are discussed at the end of Section 2.

2. A sharper theorem for semilattices

The axiomatization in Theorem 1.9 is in terms of the invertible idempotent term s on $\mathcal{M}^{[k]}$. For an arbirary mode variety, there does not seem to be any reasonable way to find these terms. However, for the special case of semilattices, the situation is brighter. Theorem 2.4 gives a straightforward way to construct all invertible, idempotent terms satisfying a condition we call irredundancy. In Lemma 2.2 we show that it is sufficient for our purposes to restrict our attention to irredundant terms. This, in conjunction with Theorem 1.9 gives a very satisfactory description of all varieties categorically equivalent to semilattices.

Lemma 2.1. [BB98, Lemma 3.1]. *Let $\mathbf{A}$ be an algebra, k a positive integer, and s an invertible idempotent term for $\mathbf{A}^{[k]}$. Suppose there exist an integer $m < k$ and terms $u_{m+1}, \dots, u_k \in \mathrm{Clo}_m(\mathbf{A})$ such that for each $\bar{b} = (b_1, \dots, b_k) \in s(A^k)$*

$$\bar{b} = (b_1, \dots, b_m,\, u_{m+1}(b_1, \dots, b_m), \dots, u_k(b_1, \dots, b_m)).$$

Then there is an invertible idempotent term s' for $\mathbf{A}^{[m]}$ such that $\mathbf{A}^{[k]}(s) \equiv_w \mathbf{A}^{[m]}(s') \equiv_c \mathbf{A}$.

It is not hard to see that if two algebras are weakly isomorphic then the varieties they generate will be term-equivalent, see [MMT87, Theorem 4.140].

Lemma 2.2. *Let $\mathcal{V}$ be a finitely generated variety, k a positive integer, and $s = (s_1, \dots, s_k)$ an invertible idempotent term on $\mathcal{V}^{[k]}$. Suppose that for some $i \le k$ there is an $(k-1)$-ary term u of $\mathcal{V}$ such that $\mathcal{V}$ satisfies the identity*

$$s_i(\bar{x}) = u(s_1(\bar{x}), s_2(\bar{x}), \dots, s_{i-1}(\bar{x}), s_{i+1}(\bar{x}), \dots, s_k(\bar{x}))$$

(where $\bar{x} = (x_1, \dots, x_k)$). Then there is an invertible idempotent term s' on $\mathcal{V}^{[k-1]}$ such that $\mathcal{V}^{[k]}(s) \equiv_t \mathcal{V}^{[k-1]}(s')$.

Proof. Let $\mathbf{A}$ be a finite generator of $\mathcal{V}$. By rearranging the components of s, we can assume that $i = k$. For any $\bar{b} \in s(A^k)$, there is $\bar{a} \in A^k$ such that $\bar{b} = s(\bar{a})$. Then in the k^{th} component, we have

$$b_k = s_k(\bar{a}) = u(s_1(\bar{a}), \dots, s_{k-1}(\bar{a})) = u(b_1, \dots, b_{k-1}).$$

Thus, the condition in Lemma 2.1 obtains with $m = k - 1$. □

We shall call an idempotent term *redundant* if an identity as in Lemma 2.2 holds. Otherwise, s shall be called irredundant. Let $\mathbf{F}$ denote the free $\mathcal{V}$-algebra on generators $x_1, \dots, x_k$. Then redundance can be recast as follows: s is redundant in $\mathcal{V}$ if for some $i < k$, s_i lies in the subalgebra of $\mathbf{F}$ generated by $\{s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_k\}$. From Lemma 2.2 we see that it suffices, in McKenzie's Theorem, to restrict our attention to irredundant, invertible, idempotent terms.

In what follows, $\mathcal{S}$ shall denote the variety of semilattices, with operation symbol '$\vee$'. For each positive integer k, let $N_k = \{1, 2, 3, \dots, k\}$. Also, $\mathbf{F}$ shall denote the free semilattice on the generators $x_1, \dots, x_k$. In the context of semilattices, the idempotent term $s = (s_1, \dots, s_k)$ on $\mathcal{S}^{[k]}$ is redundant if, for some $i \le k$, there are $j_1, \dots, j_m \le k$ with $i \notin \{j_1, \dots, j_m\}$ and $s_i = s_{j_1} \vee s_{j_2} \vee \dots \vee s_{j_m}$ in $\mathbf{F}$.

Definition 2.3. Let k be a positive integer and α a partial ordering of N_k. For each $1 \le i \le k$, we define the k-ary semilattice term

$$s_i^\alpha = \bigvee_{(j,i) \in \alpha} x_j.$$

Let $s^\alpha = (s_1^\alpha, \dots, s_k^\alpha)$. Thus s^α is a unary term on $\mathcal{S}^{[k]}$.

Theorem 2.4. *Let k be a positive integer. For every partial ordering α of N_k, s^α is an irredundant, invertible, idempotent term on $\mathcal{S}^{[k]}$. Conversely, every irredundant, invertible idempotent term on $\mathcal{S}^{[k]}$ is of the form s^α for some partial order α.*

Proof. Let α be an ordering of N_k and let $s = s^\alpha$ be the corresponding term on $\mathcal{S}^{[k]}$. Then for each $i \le k$, the identity $s_i(s_1, \dots, s_k) = s_i$ follows immediately from the definition. Hence, s is idempotent, and by Corollary 1.5, s is invertible. Suppose that s fails to be irredundant. Then for some $i, j_1, \dots, j_m \le k$, the identity $s_i(x_1, \dots, x_k) = \bigvee_{\ell=1}^m s_{j_\ell}(x_1, \dots, x_k)$ holds in $\mathcal{S}$ and $i \notin \{j_1, \dots, j_m\}$. Working in $\mathbf{F}$, $x_i \le s_i$, so for some $\ell \le m$, $x_i \le s_{j_\ell}$, hence $(i, j_\ell) \in \alpha$. On the other hand, $x_{j_\ell} \le s_{j_\ell} \le s_i$, so $(j_\ell, i) \in \alpha$. We conclude that $i = j_\ell$, which is a contradiction.

Conversely, let $s = (s_1, \dots, s_k)$ be an arbitrary unary term of $\mathcal{S}^{[k]}$. For each $i \le k$, s_i is a k-ary semilattice term in the variables $x_1, \dots, x_k$. Let D_i denote the set of those $j \le k$ such that s_i depends on the variable x_j. It is helpful to think of these terms as elements of **F** as we did just above. In **F** we have

$$D_i = \{\, j : s_i \geq x_j \,\} \quad \text{and}$$
$$s_i = \bigvee_{j \in D_i} x_j. \tag{8}$$

Now suppose that s is idempotent and irredundant. Then for every $i \leq k$, $s_i(s_1, \ldots, s_k) = s_i$ is an identity of $\mathcal{S}$. Thus, in the free semilattice **F** we have

$$s_i = \bigvee_{j \in D_i} s_j. \tag{9}$$

It follows from irredundance that $i \in D_i$. Moreover, for all $i, j \leq k$, we have

$$j \in D_i \iff s_j \leq s_i. \tag{10}$$

One direction of this follows immediately from equation (9). In the other direction, since $j \in D_j$, equations (8) imply that $x_j \leq s_j \leq s_i$, so $j \in D_i$.

Therefore, if we define $\alpha = \{\, (j,i) : s_j \leq s_i \,\}$ then α is a partial ordering; in fact, the poset $\langle\, N_k, \alpha \,\rangle$ is isomorphic to the ordering on $\{s_1, \ldots, s_k\}$ in **F** under the mapping $i \mapsto s_i$. The equality $s^\alpha = s$ now follows from Definition 2.3, equations (9) and (10) and the definition of α. □

Thus we can generate all idempotent, irredundant terms (hence all varieties categorically equivalent to $\mathcal{S}$) by generating all finite posets. To complete this analysis, we will show in Theorem 2.8 that non-isomorphic posets give rise to varieties that are not term-equivalent. We need several lemmas. These are surely known to others (see for example [DW83, Theorem 4.10] or [Lar95, Lemma 2.1] for the first, and [Fre66] for the second). However, we have not been able to find in the literature the precise formulation that we need here.

For any collection T of operations on a set A, let T^* denote the set of all operations on A that commute with every member of T. T^* is often called the *centralizer* of T. It is easy to see that T^* is always a clone on A. Let $\mathbf{2} = \langle\, \{0,1\}, \vee \,\rangle$ denote the two-element join-semilattice.

Lemma 2.5. *The clone of term operations on* **2** *is equal to* $\{\vee, 0, 1\}^*$. (*Here the symbols* '0' *and* '1' *do double-duty as elements of the semilattice and as the constant operations with values* 0 *and* 1 *respectively.*)

Proof. Since the join operation is idempotent and commutes with itself, we have $\mathrm{Clo}(\mathbf{2}) \subseteq \{\vee, 0, 1\}^*$. Suppose conversely that the n-ary operation f lies in the centralizer. Since f must preserve both 0 and 1, f is not constant. Hence $n > 0$. Since f commutes with '$\vee$', it is monotone. Without loss of generality, assume f depends on all of its variables. We wish to show that $f(x_1, \ldots, x_n) = x_1 \vee x_2 \vee \ldots \vee x_n$.

Suppose not. Since f preserves 0, $f(0, 0, \ldots, 0) = 0$. Therefore, there exist $a_1, \ldots, a_n$, not all 0, such that $f(a_1, \ldots, a_n) = 0$. For simplicity, let us assume that $a_1 = 1$. By monotonicity, we have $f(1, 0, 0, \ldots, 0) = 0$. Since f depends on its first variable, there are $b_2, b_3, \ldots, b_n \in \{0, 1\}$ such that $f(0, b_2, \ldots, b_n) \neq f(1, b_2, \ldots, b_n)$. But since f commutes with '$\vee$',

$$f(1, b_2, \ldots, b_n) = f(1, 0, 0, \ldots, 0) \vee f(0, b_2, \ldots, b_n) = 0 \vee f(0, b_2, \ldots, b_n) = f(0, b_2, \ldots, b_n),$$

a contradiction. □

Lemma 2.6. *Let '+' and '$*$' be binary operations on a set A, each with a two-sided identity element . If $+$ and $*$ commute with each other, then they are equal.*

Proof. Let 0 and 1 be the identity elements for $+$ and $*$ respectively. The assertion that $+$ and $*$ commute is equivalent to the identity $(x+u)*(v+y) = (x*v)+(u*y)$. Taking $u = v = 0$ we obtain

$$x * y = (x * 0) + (0 * y). \tag{11}$$

Setting $x = y = 1$ in this new equation yields

$$1 = 1 * 1 = (1 * 0) + (0 * 1) = 0 + 0 = 0.$$

Therefore, equation (11) becomes $x * y = x + y$. □

Recall our definition of $f^{[k]}$ for an operation f on a set A (Definition 1.3). The next lemma requires a straightforward, but tedious, verification. The details are worked out in [DL95].

Lemma 2.7. *Let $\mathbf{A}$ be an algebra and T a set of operations on A. Suppose that $\mathrm{Clo}(\mathbf{A}) = T^*$. If s is an invertible idempotent term on $\mathbf{A}^{[k]}$, then $\mathrm{Clo}(\mathbf{A}^{[k]}(s)) = \{\, (f^{[k]})_s : f \in T \,\}^*$.*

Theorem 2.8. *Let k and ℓ be positive integers. If α and β are partial orders on N_k and N_ℓ respectively, then $\mathcal{S}^{[k]}(s^\alpha) \equiv_t \mathcal{S}^{[\ell]}(s^\beta)$ if and only if $\langle N_k, \alpha \rangle \cong \langle N_\ell, \beta \rangle$.*

Proof. Suppose that α and β are partial orders on N_k and N_ℓ respectively. Certainly if $\langle N_k, \alpha \rangle \cong \langle N_\ell, \beta \rangle$ then $\mathcal{S}^{[k]}(s^\alpha) \equiv_t \mathcal{S}^{[\ell]}(s^\beta)$ since the term s^β will simply be a rearrangement of the components of s^α. So now we suppose that $\mathcal{S}^{[k]}(s^\alpha) \equiv_t \mathcal{S}^{[\ell]}(s^\beta)$.

Let s be an invertible idempotent term on $\mathcal{S}^{[k]}$. By Lemmas 2.5 and 2.7, $\mathrm{Clo}(\mathbf{2}^{[k]}(s)) = \{\vee_s^{[k]}, 0_s^{[k]}, 1_s^{[k]}\}^*$, and furthermore, it is easy to check that $\vee_s^{[k]}$ is a semilattice operation on $s(2^k)$ with bounds $s(0, 0, \ldots, 0) = 0^{[k]}$ and $s(1, \ldots, 1) = 1^{[k]}$. Since $\mathbf{2}$ is the unique simple algebra in $\mathcal{S}$, it follows that $\mathbf{2}^{[k]}(s)$ is the unique simple algebra in $\mathcal{S}^{[k]}(s)$. Thus the term equivalence of $\mathcal{S}^{[k]}(s^\alpha)$ and $\mathcal{S}^{[k]}(s^\beta)$ implies that the algebras $\mathbf{2}^{[k]}(s^\alpha)$ and $\mathbf{2}^{[k]}(s^\beta)$ are weakly isomorphic.

Therefore, there are two semilattice operations, $\vee_1$ and $\vee_2$ on $s^\alpha(2^k)$ (one equal to $\vee_{s^\alpha}^{[k]}$, the other obtained from $\vee_{s^\beta}^{[\ell]}$ via the weak isomorphism) with bounds a_i and b_i (for $i = 1, 2$) such that $\mathrm{Clo}(\mathbf{2}^{[k]}(s^\alpha)) = \{\vee_1, a_1, b_1\}^* =$

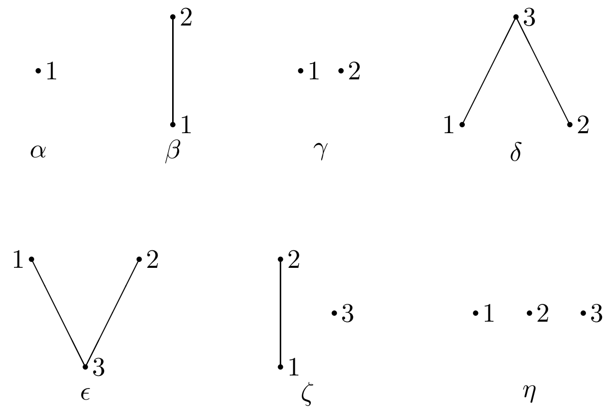

FIGURE 2

$\{\vee_2, a_2, b_2\}^*$. Since $\vee_1$ commutes with itself (and a_1 and b_1), it must commute with $\vee_2$. Therefore, by Lemma 2.6, the two join operations coincide. Consequently, the semilattices $\langle s^\alpha(2^k), \vee^{[k]}_{s^\alpha}\rangle$ and $\langle s^\beta(2^\ell), \vee^{[\ell]}_{s^\beta}\rangle$ are isomorphic. The conclusion $\langle N_k, \alpha\rangle \cong \langle N_\ell, \beta\rangle$ now follows from the following claim.

Claim. *For any partial ordering α of N_k, the poset $\langle N_k, \alpha\rangle$ is isomorphic to the dual of the poset of join-irreducible elements of the semilattice $\langle s^\alpha(2^k), \vee^{[k]}_\alpha\rangle$.*

Proof of Claim. Let us write s in place of s^α. Let $e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ (with a '1' in the i^{th} position), for $i = 1, 2, \ldots, k$. We shall demonstrate that the mapping $i \mapsto s(e_i)$ provides the desired dual isomorphism. First, it is easy to verify from Definition 2.3 that for any $\bar{b} = (b_1, \ldots, b_k) \in s(2^k)$ we have $\bar{b} = \bigvee_{b_i=1} s(e_i)$. Consequently, the only candidates for join-irreducible elements are $s(e_1), \ldots, s(e_k)$.

Now, observe that for any $i, j \in N_k$,

$$(j, i) \in \alpha \iff s_i(e_j) = 1.$$

To see that our assignment is order-reversing, let $(j, i) \in \alpha$. Then, for any $\ell \leq k$, $s_\ell(e_i) = 1 \implies (i, \ell) \in \alpha \implies (j, \ell) \in \alpha \implies s_\ell(e_j) = 1$. Thus $s(e_i) \leq s(e_j)$. Similarly, the map is one-to-one, for if $(j, i) \notin \alpha$, then $s_i(e_i) = 1$, but $s_i(e_j) = 0$.

Finally, suppose that $s(e_i) = s(e_{j_1}) \vee \ldots \vee s(e_{j_m})$. Since $s_i(e_i) = 1$, there is some $\ell \leq m$ such that $s_i(e_{j_\ell}) = 1$, hence $(j_\ell, i) \in \alpha$, so $s(e_i) \leq s(e_{j_\ell})$. Therefore, $s(e_i) = s(e_{j\ell})$, in other words, $s(e_i)$ is join-irreducible. □

Putting all of this together, we obtain a very satisfactory procedure for enumerating all varieties (up to term-equivalence) that are categorically equivalent to $\mathcal{S}$. For each finite poset P with $|P| = k$ (up to isomorphism), use 2.3 to build a unary term s on $\mathcal{S}^{[k]}$, which, by 2.4 will be idempotent and invertible. Then 1.9 gives an axiomatization of $\mathcal{S}^{[k]}(s)$.

For example, the posets of cardinality at most 3 are given in Figure 2. Of course $s^\alpha = (x)$ and $\mathcal{S}^{[1]}(s^\alpha) \equiv_t \mathcal{S}$. For the poset $\langle N_2, \beta\rangle$, we have $s^\beta = (x,\, x \vee y)$. The variety $\mathcal{S}^{[2]}(s^\beta)$ can be described, much as we did in Examples 1.11 and 1.12 with two binary and two unary operations satisfying the following identities:

$$
\begin{aligned}
&x \vee x = x\\
&x \vee y = y \vee x\\
&(x \vee y) \vee z = x \vee (y \vee z)\\
&t_1t_2(x) = t_2(x), \quad t_2t_1(x) = x\\
&t_1(x) * t_2(x) = x\\
&t_1(x \vee y) = t_1(x) \vee t_1(y), \quad t_2(x \vee y) = t_2(x) \vee t_2(y)\\
&t_1(x * y) = t_1(x)\\
&t_2(x * y) = t_1(x) \vee t_2(y).
\end{aligned}
$$

Put another way, this is the subvariety of $\mathcal{E}_2 \otimes \mathcal{S}$ defined by the last two of these identities. The smallest nontrivial algebra in this variety is $\mathbf{2}^{[2]}(s^\beta)$, which can be taken to have universe $\{a, b, c\}$ and join-semilattice structure such that $a < b < c$. In fact, $a = (0, 0)$, $b = (0, 1)$ and $c = (1, 1)$. The tables for the other operations are

$*$	a	b	c
a	a	b	b
b	a	b	b
c	c	c	c

	t_1	t_2
a	a	a
b	a	c
c	c	c

The posets γ and η yield the second and third matrix powers of $\mathcal{S}$. So let us turn to δ. The idempotent term $s^\delta = (x,\, y,\, x \vee y \vee z)$. The smallest nontrivial algebra of $\mathcal{S}^{[3]}(s^\delta)$ will have five elements. We can present this variety with operations $\vee$, d, t_1, t_2, t_3 and the identities consisting of ax1–ax4 together with

$$
\begin{aligned}
t_1d(x, y, z) &= t_1(x)\\
t_2d(x, y, z) &= t_2(y)\\
t_3d(x, y, z) &= t_1(x) \vee t_2(y) \vee t_3(z).
\end{aligned}
$$

Similarly, $s^\epsilon(x, y, z) = (x \vee z,\, y \vee z,\, z)$. The presentation for $\mathcal{S}^{[3]}(s^\epsilon)$ would be the same as the one just above, except that the last three equations become

$$
\begin{aligned}
t_1d(x, y, z) &= t_1(x) \vee t_3(z)\\
t_2d(x, y, z) &= t_2(y) \vee t_3(z)\\
t_3d(x, y, z) &= t_3(z).
\end{aligned}
$$

Finally, $s^\zeta(x, y, z) = (x, x \vee y, z)$. We leave it to the reader to determine the axiomatization and minimal nontrivial member (which has six elements).

It is interesting to compare our results to the description of varieties categorically equivalent to $\mathcal{S}$ in [DW83, 4.10]. That theorem asserts that $\mathcal{V} \equiv_c \mathcal{S}$ if and only if there is a nontrivial algebra $\mathbf{A} \in \mathcal{V}$ such that

(1) $\mathcal{V} = ISP(\mathbf{A})$, i.e., every member of $\mathcal{V}$ is isomorphic to a subalgebra of a power of $\mathbf{A}$;
(2) $\mathbf{A}$ has a binary term '$\vee$' such that $\langle A, \vee \rangle$ is a semilattice, and under the induced ordering, A forms a distributive lattice with bounds 0 and 1;
(3) $\mathrm{Clo}(\mathbf{A}) = \{\vee, 0, 1\}^*$.

We see now that for a poset $\langle N_k, \alpha \rangle$, the algebra $\mathbf{A}$ above will be $\mathbf{2}^{[k]}(s^\alpha)$, and the semilattice operation will be $\vee_{s^\alpha}^{[k]}$. The distributive lattice structure on A arises as both the order-theoretic dual of the lattice of downsets of $\langle N_k, \alpha \rangle$, and as $s(2^k)$, which turns out to be a sublattice of $\{0,1\}^k$ under the usual order.

The proof of Theorem 1.9 goes through with only minor modifications if we allow the variety $\mathcal{M}$ to have a single constant operation. All of the nonconstant operations are still required to be idempotent. This would allow us to extend our results to semilattices with identity. One interesting feature of this is that the k-ary operation d of $E(\mathcal{M}, k, s)$ could be replaced with k unary operations p_i via the definitions

$$p_i(x) = d(0, 0, \ldots, 0, x, 0, \ldots, 0) \quad i = 1, \ldots, k$$

where the lone 'x' appears in the i^{th} position of d, and 0 denotes the identity. The operation d can be recovered from $p_1, \ldots, p_k$ with

$$d(x_1, \ldots, x_k) = \bigvee_{i=1}^{k} p_i(x_i).$$

Thus, varieties categorically equivalent to semilattices with identity can be axiomatized using one binary, one nullary and several unary operations.

References

[BB98] C. Bergman and J. Berman, *Algorithms for categorical equivalence*, Math. Struc. Comp. Sci. **8** (1998), 1–15.

[DL95] K. Denecke and O. Lüders, *Category equivalence of clones*, Algebra Universalis **34** (1995), 608–618.

[DW83] B. A. Davey and H. Werner, *Dualities and equivalences for varieties of algebras*, Contributions to Lattice Theory, Colloq. Math. Soc. János Bolyai, vol. 33, North-Holland, Amsterdam, New York, 1983, pp. 101–275.

[Fre66] P. Freyd, *Algebra valued functors in general and tensor products in particular*, Colloq. Math. **14** (1966), 89–106.

[Jež82] J. Ježek, *A note on isomorphic varieties*, Commentationes Math. Univ. Carolinae **23** (1982), 579–588.

[Kea] K. Kearnes, *Subdirectly irreducible modes*, this volume.

[Lar95] B. Larose, *On the centralizer of the join operation in a finite lattice*, Algebra Universalis **34** (1995), 304–313.

[Mac71] S. Mac Lane, *Categories for the working mathematician*, Graduate Texts in mathematics, vol. 5, Springer-Verlag, New York, 1971.

[McK94] R. McKenzie, *An algebraic version of categorical equivalence for varieties and more general algebraic categories*, Lecture Notes in Pure and Applied Mathematics, vol. 180, Marcel Dekker, Inc., New York, 1994, Proceedings of the Magari Memorial Conference, pp. 211–243.

[MMT87] R. McKenzie, G. McNulty, and W. Taylor, *Algebras, lattices, varieties*, vol. I, Wadsworth & Brooks/Cole, Belmont, CA, 1987.

[PR92] J. Płonka and A. Romanowska, *Semilattice sums*, Universal Algebra and Quasigroup Theory (A. Romanowska and J. D. H. Smith, eds.), Heldermann, Berlin, 1992, pp. 123–158.

[Smi] J. D. H. Smith, *Modes and modals*, this volume.

[Tay75] W. Taylor, *The fine spectrum of a variety*, Algebra Universalis **5** (1975), 263–303.

[Wra70] G. C. Wraith, *Algebraic theories*, Aarhus University Lecture Notes Series, no. 22, Aarhus University, 1970.

Department of Mathematics, Iowa State University, Ames, Iowa, 50011–2066, USA

E-mail address: cbergman@iastate.edu

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, 851 S. Morgan (m/c 249), Chicago, Illinois 60607–7045, USA

E-mail address: jberman@uic.edu