NOTES ON QUASIVARIETIES
AND MALTSEV PRODUCTS

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These notes were prepared for my research group in 2014. These results are not new. Most appear in similar form in [4]. Since neither [4] nor its English translation, [5] is easily accessible, it seemed worthwhile to make these notes available to a wider audience. The notation and terminology follows that of my book, [1].

Definition 1. A quasivariety is a class of algebras closed under subalgebra, product, and ultraproduct. Equivalently (see [1, thm 5.4]) a class is a quasivariety iff closed under subalgebra and reduced product.

It is easy to see that the intersection of a family of quasivarieties is again a quasivariety. Thus we can talk about the quasivariety generated by a class of algebras.

Proposition 2. Let $K$ be a class of algebras. The quasivariety generated by $K$ is $\text{SPP}_u(K) = \text{SP}_r(K)$.

A proof can be found in [2, thm. V.2.23] or [1, thm. 5.4].

Corollary 3. Let $A$ be a finite algebra. The quasivariety generated by $A$ is $\text{SP}(A)$.

Definition 4. A quasiidentity is a formula of the form

\[
(p_1(x) \approx q_1(x)) \land (p_2(x) \approx q_2(x)) \land \cdots \land (p_k(x) \approx q_k(x)) \rightarrow s(x) \approx t(x)
\]

When $k = 0$ we have the identity $s(x) \approx t(x)$, so every identity is a quasiidentity.

Theorem 5. A class of algebras is a quasivariety if and only if it is defined by a set of quasiidentities.

For a proof of Theorem 5 see [2, V.2.25].

Congruence classes play a double role in the context of Maltsev products: as elements of a quotient algebra and as (potential) subalgebras. In order to help keep things straight, let us write $[a]_{\theta}$ for a congruence class being treated as a subset, and continue to write $a/\theta$ for the corresponding element of the quotient algebra.

An element, $a$, of an algebra, $A$, is called idempotent if $\{a\}$ forms a subuniverse of $A$. Put another way, for every basic operation, $f$, we have

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Figure 1

\[ f(a, a, a, \ldots, a) = a. \] The algebra \( A \) is idempotent if every element is idempotent. A class, \( \mathcal{K} \), of algebras is idempotent if every member algebra is idempotent.

**Definition 6.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be quasivarieties. Then

\[
\mathcal{A} \circ \mathcal{B} = \{ R : (\exists \theta \in \text{Con}(R)) R/\theta \in \mathcal{B} \text{ and } (\forall r \in R) [r]_{\theta} \in \text{Sub}(R) \implies [r]_{\theta} \in \mathcal{A} \}.
\]

The class \( \mathcal{A} \circ \mathcal{B} \) is called the Maltsev product of \( \mathcal{A} \) and \( \mathcal{B} \). If \( \mathcal{C} \) is another quasivariety containing both \( \mathcal{A} \) and \( \mathcal{B} \), we write \( \mathcal{A} \circ \mathcal{B} \cap \mathcal{C} = (\mathcal{A} \circ \mathcal{B}) \cap \mathcal{C} \). For the extent of this paper, by an \( \mathcal{A}, \mathcal{B} \)-pivot (or just a pivot if the context is clear) we mean a congruence \( \theta \) satisfying the conditions of Definition 6.

Let \( h : A \to B \) be a homomorphism with kernel \( \alpha \). Let \( r \in A \). Then \( [r]_{\alpha} \) is a subalgebra of \( A \) if and only if \( h(r) \) is idempotent in \( B \). This is an immediate consequence of the fact that a direct image of a subalgebra is a subalgebra and the inverse image of a subalgebra is a subalgebra. Another way to say this is

(1) \( [r]_{\alpha} \in \text{Sub}(A) \iff f(r, r, \ldots, r) \alpha r \) for every basic operation \( f \).

Now suppose that \( \alpha \leq \beta \in \text{Con}(A) \). Then

(2) \( [r]_{\alpha} \in \text{Sub}(A) \implies (\forall f) f(r, r, \ldots, r) \alpha r \implies (\forall f) f(r, r, \ldots, r) \beta r \implies [r]_{\beta} \in \text{Sub}(A) \)

in which the quantifier on \( f \) ranges over all basic operations of \( A \).

Here is another observation.

**Lemma 7.** Let \( A \) be an algebra, \( \alpha < \beta \) congruences on \( A \) and \( r \in A \).

(1) \( [r/\alpha]_{\beta/\alpha} = ([r]_{\beta})/\alpha \).

(2) \( [r]_{\beta} \in \text{Sub}(A) \iff [r/\alpha]_{\beta/\alpha} \in \text{Sub}(A/\alpha) \).

**Proof.** For (1), \( x/\alpha \in [r/\alpha]_{\beta/\alpha} \iff x/\alpha \equiv r/\alpha \mod (\beta/\alpha) \iff x \equiv r \mod (\beta) \iff x \in [r]_{\beta} \iff x/\alpha \in [r]_{\beta}/\alpha \).

The second claim follows from equivalence (1) since

\( [r]_{\beta} \in \text{Sub}(A) \iff f(r, r, \ldots, r) \beta r \iff f(r, r, \ldots, r) (\beta/\alpha) r/\alpha. \)

\[ \square \]
Let \( \mathcal{B} \) be a quasivariety and \( \mathcal{R} \) an algebra of the same similarity type as \( \mathcal{B} \). Define
\[
\Lambda^R_\mathcal{R} = \{ \theta \in \text{Con}(\mathcal{R}) : \mathcal{R}/\theta \in \mathcal{B} \}
\]
and
\[
\lambda^R_\mathcal{B} = \bigcap \Lambda^R_\mathcal{R}.
\]
The congruence \( \lambda^R_\mathcal{B} \) is called the \textit{verbal congruence on} \( \mathcal{R} \) \textit{induced by} \( \mathcal{B} \). We leave off the sub- and superscript when the context is clear. Notice that \( 1_R \in \Lambda \) since \( \mathcal{B} \) contains a trivial algebra. Observe also that
\[
\mathcal{R}/\lambda \leq \bigprod_{\theta \in \Lambda} \mathcal{R}/\theta \in \mathbf{SP}(\mathcal{B}) = \mathcal{B}.
\]
Thus \( \lambda \in \Lambda \). In fact the verbal congruence is the smallest congruence on \( \mathcal{R} \) whose induced quotient falls into the quasivariety \( \mathcal{B} \).

Now suppose that \( \mathcal{R} \in \mathcal{A} \circ \mathcal{B} \). Let \( \theta \) be any \( \mathcal{A}, \mathcal{B} \)-pivot congruence on \( \mathcal{R} \). Since \( \mathcal{R}/\theta \in \mathcal{B} \) we have \( \lambda^B \leq \theta \). Consequently, for every \( r \in \mathcal{R} \), \( [r]_\lambda \subseteq [r]_\theta \). Suppose that \( [r]_\lambda \in \text{Sub}(\mathcal{R}) \). By implication (2) \( [r]_\theta \in \text{Sub}(\mathcal{R}) \) hence \( [r]_\lambda \leq [r]_\theta \in \mathcal{A} \) which implies \( [r]_\lambda \in \mathcal{A} \). Thus, in Definition 6, we can always take the \( \mathcal{A}, \mathcal{B} \)-pivot to be \( \lambda_B \).

**Lemma 8.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be any two quasivarieties. Then \( \mathcal{A} \circ \mathcal{B} \) is closed under subalgebra.

**Proof.** Let \( \mathcal{R} \in \mathcal{A} \circ \mathcal{B} \) and let \( \theta \) be an \( \mathcal{A}, \mathcal{B} \)-pivot on \( \mathcal{R} \). Let \( \mathcal{S} \) be a subalgebra of \( \mathcal{R} \). We must show \( \mathcal{S} \in \mathcal{A} \circ \mathcal{B} \). Define \( \psi = \theta|_{\mathcal{S}} \). Then \( \psi \) is a congruence on \( \mathcal{S} \) and \( \mathcal{S}/\psi \leq \mathcal{R}/\theta \). Since \( \mathcal{B} \) is closed under subalgebras, \( \mathcal{S}/\psi \in \mathcal{B} \).

Now let \( t \in \mathcal{S} \) and assume \( [t]_\psi \in \text{Sub}(\mathcal{S}) \). We claim that \( [t]_\theta \in \text{Sub}(\mathcal{R}) \). By equivalence (1)
\[
[t]_\psi \in \text{Sub}(\mathcal{S}) \implies f(t, \ldots, t)_\psi t \implies f(t, \ldots, t)_\theta t \implies [t]_\theta \in \text{Sub}(\mathcal{R}).
\]
Finally, since \( [t]_\theta \in \text{Sub}(\mathcal{R}) \), \( [t]_\theta \in \mathcal{A} \). But \( \mathcal{A} \) is closed under subalgebras and \( [t]_\psi \leq [t]_\theta \), so \( [t]_\psi \in \mathcal{A} \) as desired. \( \square \)

**Lemma 9.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be any two quasivarieties of finite similarity type. Then \( \mathcal{A} \circ \mathcal{B} \) is closed under reduced products. If \( \mathcal{B} \) is idempotent, the requirement of finite similarity type can be dropped.

**Proof.** Let \( \mathcal{R}_i \in \mathcal{A} \circ \mathcal{B} \), for \( i \in I \), and let \( \mathcal{F} \) be a filter on \( I \). We must show \( \prod_I \mathcal{R}_i/\eta_\mathcal{F} \in \mathcal{A} \circ \mathcal{B} \). By assumption, for each \( i \in I \) we have a pivot congruence, \( \theta_i \) on \( \mathcal{R}_i \). Let us write \( \mathcal{R} = \prod_I \mathcal{R}_i \).

For every \( a, b \in \mathcal{R} \) define \( J(a, b) = \{ i \in I : (a_i, b_i) \in \theta_i \} \). Note that \( J(a, b) \supseteq [a = b] \). Let \( \psi = \{ (a, b) \in \mathcal{R}^2 : J(a, b) \in \mathcal{F} \} \). It is easy to check that \( \psi \in \text{Con}(\mathcal{R}) \) and that \( \eta_\mathcal{F} \leq \psi \). By the correspondence theorem we have \( \mathcal{R}/\psi \cong (\mathcal{R}/\eta_\mathcal{F})/(\psi/\eta_\mathcal{F}) \).

Let us write \( \mathcal{R} \) in place of \( \mathcal{R}/\eta_\mathcal{F} \), \( \bar{\psi} \) for \( \psi/\eta_\mathcal{F} \) and \( \bar{r} \) in place of \( r/\eta_\mathcal{F} \). Then the isomorphism in the previous paragraph can be rewritten as \( \mathcal{R}/\psi \cong \mathcal{R}/\bar{\psi} \).

Our task is to show that \( \mathcal{R} \in \mathcal{A} \circ \mathcal{B} \). \( \bar{\psi} \) will be the pivot congruence on \( \mathcal{R} \) that makes this happen.
Let $h$ be the composite of the natural maps $R \to \prod (R_i/\theta_i) \to \prod (R_i/\theta_i)/\eta F$. Then $h$ is surjective and unwinding the definition shows that $\ker(h) = \psi$. Thus

\begin{equation}
\mathbf{R}/\bar{\psi} \cong R/\psi \cong \prod (R_i/\theta_i)/\eta F \in \mathcal{B}
\end{equation}

since $\mathcal{B}$ is closed under reduced products.

Now let $\bar{r} \in \mathbf{R}$ and suppose that $[\bar{r}]_\psi$ is a subuniverse of $\mathbf{R}$. We must show that $[\bar{r}]_\psi \in \mathcal{A}$. Let $r$ be an element of $R$ such that $r/\eta F = \bar{r}$. Note that $r$ is not unique. By Lemma 7, $[\bar{r}]_\psi = [r]_\psi/\eta F$ and $[r]_\psi \leq R$.

Claim: Let $K = \{ i \in I : [r_i]_{\theta_i} \in \text{Sub}(R_i) \}$. Then $K \in \mathcal{F}$.

Proof: First, if $\mathcal{B}$ is idempotent then $K = I$ which is automatically a member of $\mathcal{F}$. Now assume that the similarity type consists of finitely many operation symbols $f_1, \ldots, f_m$. Then for any $i \in I$, the condition that $[r_i]_{\theta_i}$ be a subuniverse is equivalent to

\[
(f_1(r, r, \ldots, r) \theta_i r) \& (f_2(r, r, \ldots, r) \theta_i r) \& \cdots \& (f_m(r, r, \ldots, r) \theta_i r)
\]

which in turn is equivalent to

\[
i \in J(f_1(r, \ldots, r), r) \cap J(f_2(r, \ldots, r), r) \cap \cdots \cap J(f_m(r, \ldots, r), r).
\]

But $[r]_\psi$ is a subuniverse, so for each $j \leq m$, $J(f_j(r, \ldots, r), r) \in \mathcal{F}$. Hence $K = \bigcap_{j=1}^{m} J(f_j(r, \ldots, r), r) \in \mathcal{F}$.

Let $\mathcal{F}' = \{ X \cap K : X \in \mathcal{F} \}$. Then one easily checks that $\mathcal{F}'$ is a filter on $K$. We shall show that

\begin{equation}
[r]_\psi/\eta F \cong \prod_{k \in K} [r_k]_{\theta_k}/\eta F'
\end{equation}

This will finish the proof since for $k \in K$, $[r_k]_{\theta_k} \in \text{Sub}(R_k)$, hence by assumption, $[r_k]_{\theta_k} \in \mathcal{A}$. Thus $[\bar{r}]_\psi = [r]_\psi/\eta F \in \mathcal{P}(\mathcal{A}) \subseteq \mathcal{A}$.

Recall that if $x \in [r]_\psi$ then $J(x, r) \in \mathcal{F}$, hence $J(x, r) \cap K \subseteq \mathcal{F}'$. For such an $x$, define, for each $k \in K$

\[
\bar{x}_k = \begin{cases} x_k & \text{if } k \in J(x, r), \\ r_k & \text{otherwise}. \end{cases}
\]

Notice that $\bar{x} \in \prod_{k \in K} [r_k]_{\theta_k}$ and $\bar{x}$ agrees with $x$ in “almost all” components.

Now define the map $g : [r]_\psi \to \prod_{k \in K} [r_k]_{\theta_k}/\eta F'$ by

\[
g(x) = \bar{x}/\eta F'.
\]

$g$ is easily seen to be a surjective homomorphism. We can finish the verification of (4) by showing that $\ker(g) = \eta F$ on $[r]_\psi$. So let $x, y \in [r]_\psi$. Then $x \psi y$ implies $J(x, y) \in \mathcal{F}$. Let $Z = \{ k \in K : \bar{x}_k = \bar{y}_k \}$. Then

\[
g(x) = g(y) \iff Z \in \mathcal{F}' \iff K \cap J(x, y) \cap Z \in \mathcal{F}.
\]

But $K \cap J(x, y) \cap Z \subseteq [x = y]$, so $[x = y] \in \mathcal{F}$, hence $(x, y) \in \eta F$ as desired. \qed
Theorem 10. The Maltsev product of two quasivarieties of finite type is again a quasivariety. (If the second quasivariety is idempotent, the assumption of finite type can be dropped.)


Lemma 11. If $\mathcal{A}$ and $\mathcal{B}$ are idempotent quasivarieties, then $\mathcal{A} \circ \mathcal{B}$ is idempotent.

Proof. Let $R \in \mathcal{A} \circ \mathcal{B}$ and $r \in R$. We must show that $r$ is idempotent. Let $\theta$ be a pivot congruence on $R$. Since $\mathcal{B}$ is idempotent, $r/\theta$ is an idempotent element of $R/\theta \in \mathcal{B}$, so $[r]_{\theta}$ is a subuniverse of $R$. Hence $[r]_{\theta} \in \mathcal{A}$. Since all members of $\mathcal{A}$ are idempotent and $r \in [r]_{\theta}$, $r$ is an idempotent element. □

The noteworthy thing about idempotence is that every congruence class is a subuniverse. Thus when both $\mathcal{A}$ and $\mathcal{B}$ are idempotent, we can ignore the clause “[r]_{\theta} \in \text{Sub}(R)” in the definition of Maltsev product.

Assume that $\mathcal{A}$ and $\mathcal{B}$ have finite similarity type, or that $\mathcal{B}$ is idempotent. Then $\mathcal{C} = \mathcal{A} \circ \mathcal{B}$ is a quasivariety, by Theorem 10. Let $F = F_\mathcal{C}(X)$ be a free $\mathcal{C}$-algebra over a set $X$. Then $F/\lambda^\mathcal{F}_\mathcal{B} \cong F_\mathcal{B}(X)$, the free $\mathcal{B}$-algebra on $X$, [1, thm. 4.28]. Since $\lambda_\theta$ can always serve as a pivot, we must have $[r]_\lambda \in \text{Sub}(F) \implies [r]_\lambda \in \mathcal{A}$. Unfortunately, there does not seem to be a natural way to view the algebra $[r]_\lambda$ as a homomorphic image of a free algebra on $\mathcal{A}$.

As a rule, the Maltsev product of two varieties need not be a variety (even in the idempotent case). However, if all congruences permute then we do indeed get a variety.

Theorem 12. Let $\mathcal{A}$ and $\mathcal{B}$ be idempotent subvarieties of a quasivariety $\mathcal{C}$, and suppose that $\mathcal{C}$ is congruence-permutable (see [1, pg. 122]). Then $\mathcal{A} \circ \mathcal{C} \circ \mathcal{B}$ is a variety.

Proof. By Theorem 10, we already know that the Maltsev product is closed under subalgebra and product, so the only thing left to show is closure under homomorphic images. For this let $R \in \mathcal{A} \circ \mathcal{B}$ and $\alpha \in \text{Con}(R)$. We must show $R/\alpha \in \mathcal{A} \circ \mathcal{B}$. Let $\theta$ be an $\mathcal{A},\mathcal{B}$-pivot on $R$.

Let $\hat{\theta} = \theta \vee \alpha = \theta \circ \alpha$ (by congruence-permutability). We wish to show that $\hat{\theta}$ is an $\mathcal{A},\mathcal{B}$-pivot on $R/\alpha$, that is

\begin{align}
(5) & \quad (R/\alpha)/(\hat{\theta}/\alpha) \in \mathcal{B} \quad \text{and} \\
(6) & \quad r \in R \implies [r/\alpha]_{\hat{\theta}/\alpha} \in \mathcal{A}.
\end{align}

Note that we are tacitly appealing to idempotence in the formulation of (6). The first of these is easy. By the second isomorphism theorem [1, thm. 3.5],

\begin{align}
(R/\alpha)/(\hat{\theta}/\alpha) & \cong R/\hat{\theta} \in \text{H}(R/\theta) \subseteq \mathcal{B}.
\end{align}

Now let $r \in R$ and set $A = [r]_{\theta}$. $A$ is a subalgebra of $R$ by idempotence and $A \in \mathcal{A}$ by assumption. Define

\[ A^\alpha = \bigcup_{a \in A} [a]_\alpha. \]
By the third isomorphism theorem [1, thm. 3.8]
\[ \mathbf{A}^\alpha / (\alpha |_{\mathbf{A}^\alpha}) \cong \mathbf{A} / \alpha |_{\mathbf{A}} \in \mathbf{A}. \]
However, \( \mathbf{A}^\alpha = [r]_\theta \) since by congruence permutability
\[ x \in \mathbf{A}^\alpha \iff (\exists a \in R) x \alpha a \theta r \iff x \bar{\theta} r \iff x \in [r]_\bar{\theta}. \]
Finally, to verify (6) we need only observe that \([r/\alpha]_{\bar{\theta}/\alpha} = [r]_{\bar{\theta}/\alpha} = \mathbf{A}^\alpha/\alpha \).
\[ \square \]

**Example 13** (Li, 2017). Let \( CIB \) denote the variety of all commutative, idempotent binars, and let \( \mathbf{Sq} \) be the variety of binars satisfying the identities
\[ (7) \quad x^2 \approx x, \quad xy \approx yx, \quad x(xy) \approx y. \]
This is the variety of squags. Let \( q(x,y,z) = y(xz) \). Then it is easy to check that \( q \) is a Maltsev term for \( \mathbf{Sq} \) [1, thm. 4.64]. Now define the term
\[ p(x,y,z) = (x(z(xy))) \cdot (z(x(zy))). \]
Then \( p \) is a Maltsev term for \( \mathbf{Sq} \circ \mathbf{Sq} \).

**Proof.** Let \( \mathbf{A} \in \mathbf{Sq} \circ \mathbf{Sq} \). Thus, there is \( \theta \in \text{Con}(\mathbf{A}) \) such that \( \mathbf{A}/\theta \in \mathbf{Sq} \) and every \( x/\theta \in \mathbf{Sq} \).

We shall show that \( \mathbf{A} \models p(x,x,z) \approx z \), i.e., \( (x(\bar{\theta} x)))(x(\bar{\theta} z)) \approx z \). Let \( w = x(\bar{\theta} x) \). Since \( \mathbf{A}/\theta \in \mathbf{Sq} \),
\[ w/\theta = x/\theta \cdot (z/\theta \cdot x/\theta) = z/\theta \]
thus \( w, z \in [z]_{\theta} \in \mathbf{Sq} \). But then (working in \( [z]_{\theta} \)) \( p(x,x,z) \approx w(zw) \approx z \) as desired. The other identity, \( p(x,z,z) \approx x \), is similar. \[ \square \]

Thus, by Theorem 12, \( \mathbf{Sq} \circ \mathbf{Sq} \) is a variety. (Take \( \mathcal{A} = \mathcal{B} = \mathbf{Sq} \) and \( \mathcal{C} = \mathbf{Sq} \circ \mathbf{Sq} \).)

It would be interesting to find an equational base for \( \mathbf{Sq} \circ \mathbf{Sq} \).

**References**

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