JOINS AND MALTSEV PRODUCTS OF CONGRUENCE PERMUTABLE VARIETIES

CLIFFORD BERGMAN

Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be idempotent varieties and suppose that the variety $\mathcal{A} \vee \mathcal{B}$ is congruence permutable. Then the Maltsev product $\mathcal{A} \circ \mathcal{B}$ is also congruence permutable.

A group, $\mathbf{G}$, is called an extension of $\mathbf{A}$ by $\mathbf{B}$ if there is a normal subgroup, $\mathbf{N}$, of $\mathbf{G}$, such that $\mathbf{N} \cong \mathbf{A}$ and $\mathbf{G}/\mathbf{N} \cong \mathbf{B}$. In a series of papers, Bernard and Hanna Neumann explored the properties of the class $\mathcal{A}\mathcal{B}$ of groups, each of which is an extension of a member of the class $\mathcal{A}$ by a member of the class $\mathcal{B}$. They restricted their attention to the case that both $\mathcal{A}$ and $\mathcal{B}$ are varieties of groups (and $\mathcal{A}\mathcal{B}$ is defined to consist only of groups). Among other things, they proved that $\mathcal{A}\mathcal{B}$ is again a variety, that $(\mathcal{A}\mathcal{B})\mathcal{C} = \mathcal{A}(\mathcal{B}\mathcal{C})$, and that $\mathcal{A}\mathcal{B}$ is locally finite if both $\mathcal{A}$ and $\mathcal{B}$ are locally finite. They also described the full set of equations that hold in $\mathcal{A}\mathcal{B}$ in terms of those that hold in $\mathcal{A}$ and $\mathcal{B}$. A full accounting of their results can be found in [9, Chap. 2].

In [7] A. I. Maltsev considered this construction in a very general context. Among his observations, he showed that if $\mathcal{A}$ and $\mathcal{B}$ are quasivarieties of finite similarity type, then the Maltsev product, which we denote $\mathcal{A} \circ \mathcal{B}$, is again a quasivariety. Moreover, this product contains both $\mathcal{A}$ and $\mathcal{B}$. Consequently, $\mathcal{A} \vee \mathcal{B} \subseteq \mathcal{A} \circ \mathcal{B}$ (join in the lattice of quasivarieties.) We reproduce Maltsev's construction, specifically for quasivarieties, in Definition 1.1 below.

Unfortunately, (and in contrast to the situation for groups), it is not the case that the Maltsev product of two varieties be closed under homomorphic images. To address this failure, Maltsev introduced a further restriction by requiring his algebras to be *polarized.* A class, $\mathcal{C}$, is polarized if there is a basic unary operation symbol that is constant on every member of $\mathcal{C}$, and that constant is an idempotent element of the algebra. This constant is called the *pole* of the algebra. Note that the pole of a group is its identity element, the pole of an algebra is unique (if it exists), and a congruence class of a polarized algebra is a subalgebra if and only if it is the congruence class of the pole. Maltsev proved that if $\mathcal{C}$ is a congruence-permutable, polarized variety, then, for any two subvarieties, $\mathcal{A}$ and $\mathcal{B}$, the class $(\mathcal{A} \circ \mathcal{B}) \cap \mathcal{C}$ is again a variety.

Date: February 15, 2018.

Recently, interest in universal algebra has turned in a somewhat different direction, towards idempotent algebras. It is easy to see that the Maltsev product of two idempotent quasivarieties is again idempotent. Freese and McKenzie, [4] consider the preservation of various properties under the product. While they show that a number of important Maltsev conditions are preserved, congruence-permutability is not one of them.

In this short paper we provide some context for this failure. The main result shows that for idempotent varieties $\mathcal{A}$ and $\mathcal{B}$, if $\mathcal{A} \vee \mathcal{B}$ is congruence-permutable, then so is $\mathcal{A} \circ \mathcal{B}$. Combining this with Maltsev's argument described above, if $\mathcal{A} \vee \mathcal{B}$ is congruence-permutable, then $\mathcal{A} \circ \mathcal{B}$ is a variety.

1. Maltsev Products

Classes of algebras are always assumed to be of some single, fixed similarity type and closed under isomorphic image. A *quasivariety* is a class closed under subalgebra, product, and ultraproduct. Equivalently, under subalgebra and reduced product. See [3, Theorem 2.25]. A quasivariety is a *variety* if it is closed under homomorphic images. For an algebra, $\mathbf{A}$, $\mathrm{Sub}(\mathbf{A})$ denotes the set of subuniverses of $\mathbf{A}$. For all other unfamiliar notions of universal algebra, consult [2].

Since we work sometimes in the lattice of varieties, and sometimes in the lattice of quasivarieties, we shall use the notation $\mathcal{A} \vee \mathcal{B}$ for the smallest variety containing $\mathcal{A} \cup \mathcal{B}$ and $\mathcal{A} \vee_{\mathsf{Q}} \mathcal{B}$ for the smallest quasivariety. Note that if $\mathcal{A}$ is a quasivariety, then its closure under homomorphic images, $\mathsf{H}(\mathcal{A})$, is the variety generated by $\mathcal{A}$.

Congruence classes play a double role in the context of Maltsev products: as elements of a quotient algebra and as (potential) subalgebras. It may be helpful to use separate notations for the congruence class of an element, a, modulo the congruence θ to distinguish these roles. We shall write $[a]_\theta$ when this congruence class is being treated as a subset, and continue to write a/θ for the corresponding element of the quotient algebra.

Definition 1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be quasivarieties. The *Maltsev product of $\mathcal{A}$ and $\mathcal{B}$* is

$$\begin{aligned}\mathcal{A} \circ \mathcal{B} = \big\{\, \mathbf{R} : (\exists \theta \in \mathrm{Con}(\mathbf{R}))\ &\mathbf{R}/\theta \in \mathcal{B} \text{ and} \\ &(\forall r \in R)\ [r]_\theta \in \mathrm{Sub}(\mathbf{R}) \implies [r]_\theta \in \mathcal{A} \,\big\}.\end{aligned}$$

If $\mathcal{A}$ and $\mathcal{B}$ are subquasivarieties of the quasivariety $\mathcal{C}$, then we write $\mathcal{A} \circ_{\mathcal{C}} \mathcal{B} = (\mathcal{A} \circ \mathcal{B}) \cap \mathcal{C}$.

An algebra is called *idempotent* if every singleton subset is a subuniverse. A class of algebras is idempotent if every member algebra is idempotent. Observe that in an idempotent algebra, every congruence class $[a]_\theta$ is a subalgebra. We summarize the basic properties of the Maltsev product in the following theorem.

Theorem 1.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be quasivarieties.*

(1) *If the similarity type is finite, or if* $\mathcal{B}$ *is idempotent, then* $\mathcal{A} \circ \mathcal{B}$ *is a quasivariety. Moreover* $\mathcal{A} \vee_{\mathbf{Q}} \mathcal{B} \subseteq \mathcal{A} \circ \mathcal{B}$.
(2) *If* $\mathcal{A}$ *and* $\mathcal{B}$ *are idempotent then* $\mathcal{A} \circ \mathcal{B}$ *is idempotent.*
(3) *If* $\mathcal{A}$ *and* $\mathcal{B}$ *are idempotent subvarieties of a congruence-permutable quasivariety,* $\mathcal{C}$*, then* $\mathcal{A} \circ_{\mathcal{C}} \mathcal{B}$ *is a variety.*

In [7] Maltsev proved 1.2(3) under the assumption that $\mathcal{C}$ is polarized rather than idempotent. However the proof is essentially the same in the idempotent case. A proof of Theorem 1.2 is also provided in [1].

2. Congruence-permutability of the Maltsev Product

In [4, Example 2.1], Freese and McKenzie exhibit idempotent varieties $\mathcal{B}_0$ and $\mathcal{B}_1$, both of which are congruence-permutable, but their join, $\mathcal{B}_0 \vee \mathcal{B}_1$ fails to be congruence-permutable. It follows from Theorem 1.2(1) that $\mathcal{B}_0 \circ \mathcal{B}_1$ can not be congruence-permutable. As we show in Theorem 2.1, this is the only obstacle to permutability of the Maltsev product.

Theorem 2.1. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be idempotent varieties. If* $\mathcal{A} \vee \mathcal{B}$ *is congruence permutable, then so is* $\mathcal{A} \circ \mathcal{B}$.

Recall that a variety (in fact, a quasivariety) is congruence-permutable if and only if there is a ternary term $q(x, y, z)$ (a Maltsev term) such that the equations $q(x, x, y) \approx q(y, x, x) \approx y$ holds. The proof of Theorem 2.1 hinges on the observation that if $\mathcal{A} \vee \mathcal{B}$ is congruence permutable, then there is a single term q that simultaneously acts as a Maltsev term on $\mathcal{A}$ and on $\mathcal{B}$. A key role is played by the following result [5, Lemma 2.8] of Kearnes and Tschantz.

Lemma 2.2. *Let* $\mathcal{W}$ *be an idempotent variety that is not congruence permutable. If* $\mathbf{F} = \mathbf{F}_{\mathcal{W}}(x, y)$ *is the 2-generated free algebra in* $\mathcal{W}$*, then* $\mathbf{F}$ *has subuniverses* U *and* V *such that*

(1) $x \in U$, $y \in V$;
(2) $y \notin U$, $x \notin V$, *and*
(3) $S = (U \times F) \cup (F \times V)$ *is a subuniverse of* $\mathbf{F} \times \mathbf{F}$.

Proof of Theorem 2.1. Let q be a Maltsev term for $\mathcal{A} \vee \mathcal{B}$. Assume $\mathcal{A} \circ \mathcal{B}$ is not congruence permutable. We shall derive a contradiction. Let $\mathcal{W} = \mathbf{H}(\mathcal{A} \circ \mathcal{B})$. Since $\mathcal{A}$ and $\mathcal{B}$ are idempotent, so is $\mathcal{W}$. Certainly $\mathcal{W}$ is not congruence permutable, so we can apply Lemma 2.2 to $\mathcal{W}$.

So set $\mathbf{F} = \mathbf{F}_{\mathcal{W}}(x, y)$. Let U and V be the subuniverses provided by the lemma, and $S = (U \times F) \cup (F \times V)$. Since $\mathbf{F}$ is free and $\mathcal{W} = \mathbf{H}(\mathcal{A} \circ \mathcal{B})$, we have $\mathbf{F} \in \mathcal{A} \circ \mathcal{B}$. Hence there is a congruence λ on $\mathbf{F}$ such that $\mathbf{G} = \mathbf{F}/\lambda \in \mathcal{B}$, $\mathbf{X} = [x]_\lambda \in \mathcal{A}$ and $\mathbf{Y} = [y]_\lambda \in \mathcal{A}$. Of course $x \in X$ and $y \in Y$.

Let $a = (x, x)$, $b = (x, y)$, $c = (y, y)$, and $d = (y, x)$. Note that $a, b, c \in S$ while $d \notin S$. We shall derive a contradiction by showing that, in fact, $d \in S$.

Let $d' = q^{\mathbf{F}^2}(a, b, c) = (p_1, p_2)$. Then $a, b, c \in S$ implies $d' \in S$ as well. From the definition of S we must have either $p_1 \in U$ or $p_2 \in V$. Without loss of generality, let us assume that

(1) $$p_2 \in V.$$

Now from the definitions of a, b, c, and d', we have $p_1 = q^{\mathbf{F}}(x, x, y)$. But $\mathbf{G} = \mathbf{F}/\lambda \in \mathcal{B}$ and q is a Maltsev term for $\mathcal{B}$, hence, $p_1/\lambda = q^{\mathbf{G}}(x/\lambda, x/\lambda, y/\lambda) = y/\lambda$, i.e., $p_1 \mathrel{\lambda} y$. Thus

(2) $$p_1 \in Y.$$

Similarly, $p_2/\lambda = q^{\mathbf{G}}(x/\lambda, y/\lambda, y/\lambda) = x/\lambda$, so

(3) $$p_2 \in X.$$

Now let $e = (x, p_2) \in U \times F \subseteq S$. Define $e' = q^{\mathbf{F}^2}(d', e, a) = (p_3, p_4)$. Then e' is a member of S as well. As before, $p_3/\lambda = q^{\mathbf{G}}(p_1/\lambda, x/\lambda, x/\lambda) = p_1/\lambda$, so

(4) $$p_3 \in Y.$$

From (3), $p_2, x \in X$, hence $p_4 = q^{\mathbf{F}}(p_2, p_2, x) = q^{\mathbf{X}}(p_2, p_2, x) = x$ since q is a Maltsev term for $\mathbf{X} \in \mathcal{A}$.

Finally, let $f_1 = (y, p_2)$ and $f_2 = (p_3, p_2)$. Then $f_1, f_2 \in F \times V \subseteq S$ by (1). Therefore $q^{\mathbf{F}^2}(f_1, f_2, e') \in S$. But

$$q^{\mathbf{F}^2}(f_1, f_2, e') = \big(\, q^{\mathbf{Y}}(y, p_3, p_3),\ q^{\mathbf{X}}(p_2, p_2, x)\,\big) = (y, x) = d$$

proving that $d \in S$. Contradiction. □

Corollary 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be idempotent varieties, and suppose that $\mathcal{A} \vee \mathcal{B}$ is congruence-permutable. Then $\mathcal{A} \circ \mathcal{B}$ is variety.*

Proof. Let $\mathcal{C} = \mathcal{A} \circ \mathcal{B}$. By Theorem 1.2(1), $\mathcal{C}$ is a quasivariety, and by Theorem 2.1, it is congruence-permutable. Therefore $\mathcal{A} \circ \mathcal{B} = \mathcal{A} \circ_{\mathcal{C}} \mathcal{B}$ is a variety by Theorem 1.2(3). □

Corollary 2.4. *Let $\mathcal{A}$ be an idempotent, congruence permutable variety. Then $\mathcal{A} \circ \mathcal{A}$ is congruence permutable. Furthermore, $\mathcal{A} \circ \mathcal{A}$ is a variety.*

Unfortunately, Theorem 2.1 does not provide a recipe for finding a Malstev term for $\mathcal{A} \circ \mathcal{B}$ given the term for $\mathcal{A} \vee \mathcal{B}$. We have managed this in one case. Let $\mathcal{S}q$ denote the variety of *squags*. This is the variety of binars defined by the identities

$$x \cdot x \approx x, \quad x \cdot y \approx y \cdot x, \quad x \cdot (x \cdot y) \approx y.$$

This variety is obviously idempotent. It is congruence-permutable, with Maltsev term $q(x, y, z) = y \cdot (x \cdot z)$. Therefore by Corollary 2.4, $\mathcal{S}q \circ \mathcal{S}q$ must be a congruence-permutable variety. In [6], Li showed that a Maltsev term for $\mathcal{S}q \circ \mathcal{S}q$ is $p(x, y, z) = \big(x(z(xy))\big) \cdot \big(z(x(zy))\big)$.

Problem. Find an equational base for $\mathcal{S}q \circ \mathcal{S}q$. Is this variety finitely based?

While we have stated Theorem 2.1 for varieties, it could just as easily have been stated for quasivarieties.

Corollary 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be idempotent quasivarieties. If $\mathcal{A} \vee_{\mathbf{Q}} \mathcal{B}$ is congruence-permutable, then $\mathcal{A} \circ \mathcal{B}$ is congruence-permutable.*

Proof. Suppose that $\mathcal{C} = \mathcal{A} \vee_{\mathbf{Q}} \mathcal{B}$ is a congruence-permutable quasivariety. Then $\mathcal{W} = \mathbf{H}(\mathcal{C})$ is a congruence-permutable variety. But it is easy to check that $\mathcal{W} = \mathbf{H}(\mathcal{A}) \vee \mathbf{H}(\mathcal{B})$. Therefore by Theorem 2.1, $\mathbf{H}(\mathcal{A}) \circ \mathbf{H}(\mathcal{B})$ is congruence-permutable. Consequently, $\mathcal{A} \circ \mathcal{B} \subseteq \mathbf{H}(\mathcal{A}) \circ \mathbf{H}(\mathcal{B})$ is congruence-permutable as well. □

Lemma 2.2 seems to be quite important in its own right. For example, we have the following very striking result. Let $\mathbf{S}_2$ denote the 2-element semilattice.

Theorem 2.6 (Kearnes)**.** *Let $\mathcal{W}$ be a variety of commutative, idempotent binars, and assume that $\mathbf{S}_2 \notin \mathcal{W}$. Then $\mathcal{W}$ is congruence-permutable.*

Proof. Assume to the contrary that $\mathcal{W}$ is not congruence-permutable. Let U and V be the subuniverses of $\mathbf{F}$ promised by Lemma 2.2. We claim that either U or V is an ideal of $\mathbf{F}$. (U is an ideal means that $u \in U$ and $a \in F$ implies $ua, au \in U$.) Suppose not. Then (because of commutativity), there are $u \in U$, $v \in V$, $a, b \in F$ such that $ua \notin U$ and $bv \notin V$. But then

$$(u, b) \cdot (a, v) = (ua, bv) \notin (U \times F) \cup (F \times V)$$

contradicting the assertion that S is a subuniverse.

Therefore, without loss of generality, we may assume that U is an ideal of $\mathbf{F}$. Then $B = U \cup \{y\}$ is a subuniverse of $\mathbf{F}$ and $\mathbf{B}$ has a congruence, θ with two blocks, namely U and $\{y\}$. Consequently, $\mathbf{S}_2 \cong \mathbf{B}/\theta \in \mathcal{W}$, a contradiction. □

References

1. Clifford Bergman, *Notes on quasivarieties and Maltsev products*, `http://www.math.iastate.edu/cbergman/manuscripts/maltsevprods.pdf`.
2. Clifford Bergman, *Universal algebra. Fundamentals and selected topics*, Pure and Applied Mathematics (Boca Raton), vol. 301, CRC Press, Boca Raton, FL, 2012. MR 2839398 (2012k:08001)
3. S. Burris and H. P. Sankappanavar, *A course in universal algebra*, Springer-Verlag, New York, 1981, Available from `http://www.math.uwaterloo.ca/~snburris/htdocs/ualg.html`.
4. Ralph Freese and Ralph McKenzie, *Maltsev families of varieties closed under join or Maltsev product*, Algebra Universalis **77** (2017), no. 1, 29–50. MR 3602782
5. Keith A. Kearnes and Steven T. Tschantz, *Automorphism groups of squares and of free algebras*, Internat. J. Algebra Comput. **17** (2007), no. 3, 461–505. MR 2333368
6. J. Li, *Congruence n-permutable varieties*, Ph.D. thesis, Iowa State University, 2017, Graduate Theses and Dissertations. 15355.
7. Anatoliĭ Ivanovič Mal′cev, *Multiplication of classes of algebraic systems*, Siberian Math. J. **8** (1967), 254–267, Translated in [8].
8. ———, *The metamathematics of algebraic systems. Collected papers: 1936–1967*, North-Holland Publishing Co., Amsterdam-London, 1971, Translated, edited, and provided with supplementary notes by Benjamin Franklin Wells, III, Studies in Logic and the Foundations of Mathematics, Vol. 66. MR 0349383

9. H. Neumann, *Varieties of groups*, Springer–Verlag, Berlin, 1967.

Department of Mathematics, Iowa State University, Ames, Iowa, 50011, USA
E-mail address: cbergman@iastate.edu