# BOOLEAN KRASNER ALGEBRAS

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An algebra $\mathbf{A}$ will be called a *group action* if the basic operations of $\mathbf{A}$ are all (unary) permutations of $A$. For the purposes of this discussion, we will disallow nullary operations. But see the remark following the proof of the Theorem.

**Theorem.** *Let $\mathbf{B}$ be a finite algebra. Then $\mathbf{B}$ is categorically equivalent to a group action if and only if for every $n$, the lattice $\mathrm{Sub}(\mathbf{B}^n)$ is Boolean.*

*Proof.* If $\mathbf{A}$ is a finite group action, then it is easy to verify that the set of subuniverses of $\mathbf{A}^n$ is closed under (set-theoretic) complementation. Consequently, $\mathrm{Sub}(\mathbf{A}^n)$ is a Boolean lattice. If $\mathbf{B} \equiv_{\mathrm{c}} \mathbf{A}$, then $\mathrm{Sub}(\mathbf{B}^n) \cong \mathrm{Sub}(\mathbf{A}^n)$, so $\mathrm{Sub}(\mathbf{B}^n)$ is Boolean as well.

Now assume that $\mathbf{B}$ has Boolean subalgebra lattices. Note that, because we are not permitting nullary operations, the empty set is the zero-element of $\mathrm{Sub}(\mathbf{B}^n)$ for every $n$. Let $\sigma$ be a unary term operation of $\mathbf{B}$ that minimizes the cardinality of $\sigma(B)$. It follows that every unary term operation is one-to-one on $\sigma(B)$. Since $B$ is finite, some power of $\sigma$ is idempotent. Thus we can assume, without loss of generality, that $\sigma$ is idempotent and every unary term operation is injective on $\sigma(B)$.

Let us show that $\sigma$ is invertible. Let $B = \{b_1, b_2, \dots, b_m\}$. We define $\mathbf{b} = \langle b_1, \dots, b_m\rangle \in B^m$, and let $\psi$ denote the subalgebra of $\mathbf{B}^m$ generated by $\{\mathbf{b}\}$. Suppose $\theta$ is an atom of $\mathrm{Sub}(\mathbf{B}^m)$ and $\theta \subseteq \psi$. Since $\theta$ is an atom, it is generated by a single $m$-tuple $\mathbf{x} = \langle x_1, \dots, x_m\rangle$. $\mathbf{x} \in \theta \subseteq \psi$ implies that there is a unary term $t$ such that $\mathbf{x} = t^{\mathbf{B}^m}(\mathbf{b})$, i.e., $x_i = t(b_i)$ for $i = 1, 2, \dots, m$. Applying $\sigma$ to both sides, $\sigma(\mathbf{x}) = \sigma(t(\mathbf{b}))$. Since $\theta$ is an atom of $\mathrm{Sub}(\mathbf{B}^m)$, it is generated by $\sigma(\mathbf{x})$.

Now since the subalgebra lattice is Boolean, every element is a join of the atoms it contains. Therefore, $\psi = \bigvee_{j=1}^{k} \theta_j$ where each $\theta_j$ is an atom. From the previous paragraph, for each $j \le k$ there is a unary term $t_j$ such that $\theta_j$ is generated by $\sigma(t_j(\mathbf{b}))$. It follows that

$$\mathbf{b} \in \psi = \bigvee_{j=1}^{k} \mathrm{Sg}^{\mathbf{B}^m}\big(\sigma(t_j(\mathbf{b}))\big) = \mathrm{Sg}^{\mathbf{B}^m}\big(\sigma(t_1(\mathbf{b})), \sigma(t_2(\mathbf{b})), \dots, \sigma(t_k(\mathbf{b}))\big).$$

Therefore, there is a term $s$ such that $\mathbf{b} = s\big(\sigma t_1(\mathbf{b}), \dots, \sigma t_k(\mathbf{b})\big)$. In other words, $\sigma$ is invertible.

Let $\mathbf{A} = \mathbf{B}(\sigma)$. We claim that $\mathbf{A}$ is a group action. Suppose that $\mathbf{A}$ had a term operation that was not essentially unary. Then there is a $k$-ary term $t_\sigma$ that depends on its first two variables. This means that there are elements $a_1, a_2, \dots, a_k, a_1' \in A$ such that

$$u = t_\sigma(a_1, a_2, \dots, a_k) \ne t_\sigma(a_1', a_2, \dots, a_k) = u'$$

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and elements $c_1, \dots, c_k, c'_2$ such that

$$v = t_\sigma(c_1, c_2, \dots, c_k) \neq t_\sigma(c_1, c'_2, \dots, c_k) = v'.$$

Now let $\delta_1 = \{\, \langle x, x, y, z\rangle : x, y, z \in B \,\}$ and $\delta_2 = \{\, \langle x, y, z, z\rangle : x, y, z \in B \,\}$. These are subalgebras of $\mathbf{B}^4$ (for any algebra $\mathbf{B}$). Each quadruple $\langle a_1, a'_1, c_1, c_1\rangle$, $\langle a_2, a_2, c_2, c'_2\rangle$, …, $\langle a_k, a_k, c_k, c_k\rangle$ is a member of either $\delta_1$ or $\delta_2$, so they are all members of $\delta_1 \vee \delta_2$. (Join in the lattice $\mathrm{Sub}(\mathbf{B}^4)$.) Therefore

$$\Big\langle \sigma t(a_1, a_2, \dots, a_k), \sigma t(a'_1, \dots, a_k), \sigma t(c_1, c_2, \dots, c_k), \sigma t(c_1, c'_2, \dots, c_k)\Big\rangle \in \delta_1 \vee \delta_2.$$

Since $t_\sigma = \sigma \circ t{\restriction}_A$, this means that $\langle u, u', v, v'\rangle \in \delta_1 \vee \delta_2$.

Let $\theta$ be the subalgebra generated by $\langle u, u', v, v'\rangle$. Then $\theta \subseteq \delta_1 \vee \delta_2$. Therefore, by distributivity, $\theta = (\theta \cap \delta_1) \vee (\theta \cap \delta_2)$. But a typical member of $\theta$ is of the form $\langle p(u), p(u'), p(v), p(v')\rangle$ for some unary term $p$ of $\mathbf{B}$. Since $p$ is injective on $\sigma(B)$, this element can not be a member of either $\delta_1$ or $\delta_2$. It follows that $\theta = \varnothing$, which is a contradiction.

Therefore $\mathbf{A}$ is term equivalent to a unary algebra. The fact that the basic operations are all permutations follows from the fact that every unary operation on $\mathbf{B}$ is injective on $\sigma(\mathbf{B})$. □

**Remark.** If we allow nullary operations, the situation seems to change somewhat. It is crucial to our argument that the zero-element of $\mathrm{Sub}(\mathbf{B}^n)$ be the empty set, a condition that fails in the presence of nullary operations. It is usually safe to replace each nullary operation with a constant unary operation. However, that has the side-effect of destroying the "Boolean" condition on the subalgebra lattices.

In the presence of nullary operation, one expects a slightly different characterization. So we conjecture: if $\mathbf{B}$ is a finite algebra such that, for every $n$, $\mathrm{Sub}(\mathbf{B}^n)$ is Boolean, then $\mathbf{B}$ is categorically equivalent to a group-action-with-constants, *i.e.,* a unary algebra in which every operation is either a permutation or constant.

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